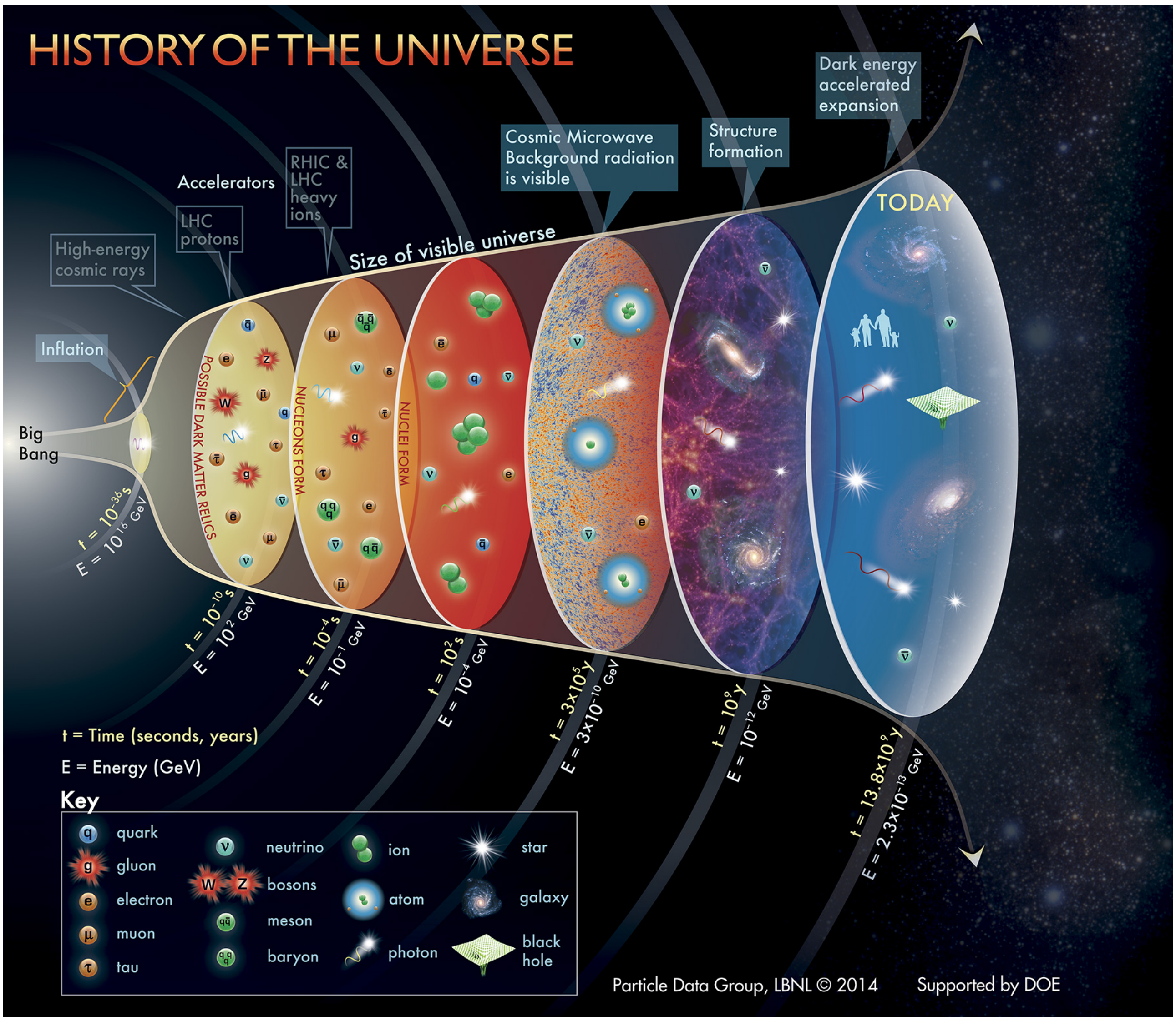
Particle Cosmology and Baryonic Astrophysics
Part I

Dr. Michael Schmidt

August 17, 2017



Contents

The first part on of the lecture will follow Dodelson, Modern Cosmology [1] closely. See also the lecture notes on cosmology by Daniel Baumann [2]. Throughout the notes, we will use natural units:

$$\hbar = c = k_B = 1 \; .$$

1 Metric and Distances

See also Dodelson, Modern Cosmology [1], chapter 2, lectures notes by Daniel Baumann [2], chapter 1, and Kolb/Turner, The Early Universe [3], chapter 1-3.

1.1 Friedmann-Robertson-Walker (FRW) Metric

Hubble discovered in 1929[4] that distant galaxies are moving away from us. His observation is shown in Fig. 1. From this diagram, we can extract the slope, called *Hubble rate* H_0, today,

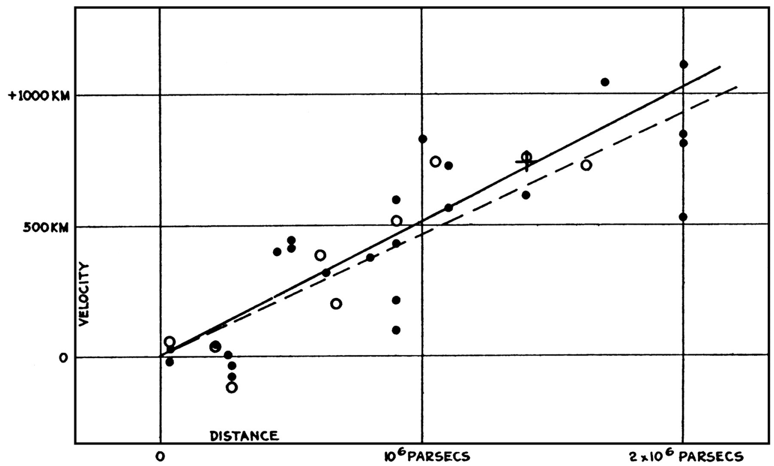


Figure 1: Hubble diagram: velocity — distance relation among extra-galactic nebulae. The velocity is in km sec^{-1} and the distance in Mpc.

$$H_0 = 100\, h \text{km}\, \text{sec}^{-1}\, \text{Mpc}^{-1}\ . \tag{1.1}$$

The Planck satellite mission measured $H_0 = (67.8 \pm 0.9)$ km sec^{-1} Mpc^{-1} [5] and thus $h = 0.68 \pm 0.9$.

Moreover when averaged over large scales the universe looks *isotropic*, i.e. the same in all directions. If we do not live at a special place, then the universe is also *homogeneous*, i.e. the same everywhere. This is commonly denoted as *cosmological principle* which motivates and indeed determines the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a(t)^2 \times d\ell^2 \qquad \text{with} \qquad d\ell^2 = \gamma_{ij} dx^i dx^j \tag{1.2}$$

where $d\ell^2$ is a symmetric 3-space:

- Euclidean space: zero curvature

$$d\ell^2 = dx^2 = \delta_{ij} dx^i dx^j \tag{1.3}$$

- S^3 (3-sphere): positive curvature

$$d\ell^2 = dx^2 + du^2 \qquad\qquad x^2 + u^2 = a^2 \tag{1.4}$$

- H^3 (3-hyperboloid): negative curvature

$$d\ell^2 = dx^2 - du^2 \qquad\qquad x^2 - u^2 = -a^2 \tag{1.5}$$

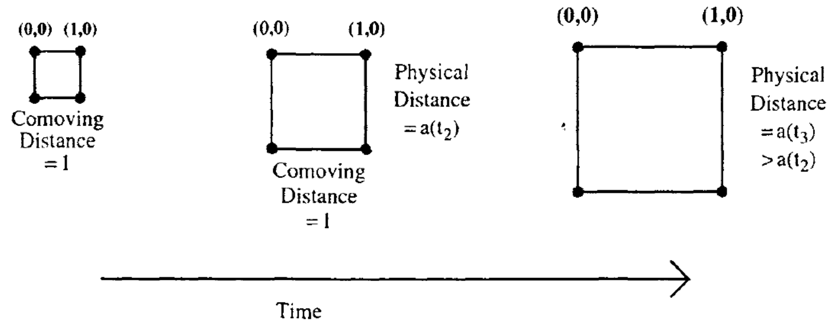


Figure 2: Expansion in an FRW Universe. Copied from [1]

After some algebra the FRW metric can be written as

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) , \tag{1.6}$$

where $k = 0$ for a flat Universe, $k = 1$ for a closed Universe with positive curvature, $k = -1$ for an open Universe with negative curvature. The parameter a is called *scale factor* and describes how the Universe expands, while k is the *curvature parameter*. See Fig. 2.

The line element is invariant under the following rescaling symmetry

$$a \to \lambda a \qquad r \to r/\lambda \qquad k \to \lambda^2 k , \tag{1.7}$$

which can be used to either set the curvature parameter to $0, \pm 1$ or the scale factor today $a(t_0) \equiv 1$. The coordinate r is called *comoving coordinate*. It is very useful in calculations. However physical results depend only on the *physical coordinate* r_{ph} and the *physical curvature* k_{ph}

$$r_{ph} = a(t) r \qquad k_{ph} = k/a^2(t) \tag{1.8}$$

The physical velocity is

$$v_{ph} \equiv \frac{dr_{ph}}{dt} = a(t)\frac{dr}{dt} + \frac{da}{dt} r \equiv v_{pec} + H r_{ph} \tag{1.9}$$

with the peculiar velocity and the *Hubble parameter*

$$H \equiv \frac{1}{a}\frac{da}{dt} \tag{1.10}$$

The metric can be conveniently rewritten in terms of a static part and the scale factor

$$ds^2 = a(t)^2 \left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) = a(t)^2 \times \text{static metric} \tag{1.11}$$

after introducing *conformal time* ("comoving time")

$$\tau = \int \frac{dt}{a(t)} . \tag{1.12}$$

Rewriting the radial component in terms of the *comoving distance*

$$\chi = \int \frac{dr}{\sqrt{1-kr^2}} \tag{1.13}$$

we obtain another useful form of the metric

$$ds^2 = a(t)^2 \left[-d\tau^2 + d\chi^2 + S_k^2(\chi)d\Omega^2\right] \tag{1.14}$$

with the *metric distance* d_m

$$d_m \equiv S_k(\chi) = \begin{cases} R_0 \sinh \frac{\chi}{R_0} & k = -1 \\ \chi & k = 0 \\ R_0 \sin \frac{\chi}{R_0} & k = +1 \end{cases} . \tag{1.15}$$

The metric distance equals the comoving distance in a flat spacetime. We will mainly consider a flat universe ($k = 0$) with metric

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & a(t)^2 & & \\ & & a(t)^2 & \\ & & & a(t)^2 \end{pmatrix} . \tag{1.16}$$

1.2 Redshift

Almost all observations rely on photons.

The wavelength of a photon similarly grows with the scale factor. The light emitted at time t_1 with wavelength λ_1 will be observed at time t_0 with wavelength[1]

$$\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1 \tag{1.17}$$

and consequently the wavelength increases when the universe is expanding, $a(t_0) > a(t_1)$ and thus redshifted. The redshift parameter z is defined as the fractional shift in wavelength

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} \tag{1.18}$$

and thus

$$1 + z = \frac{a(t_0)}{a(t_1)} . \tag{1.19}$$

Using the common definition $a(t_0) = 1$ the redshift is related to the scale factor of the emitter

$$1 + z = \frac{1}{a(t_1)} . \tag{1.20}$$

For nearby sources the scale factor can be expanded using the Hubble constant $H_0 \equiv \dot{a}(t_0)/a(t_0)$

$$a(t_1) = a(t_0)\left[1 + H_0(t_1 - t_0) + \ldots\right] \tag{1.21}$$

and thus the redshift is linearly related to the distance d of the light source

$$z \simeq H_0 d . \tag{1.22}$$

[1] This can be derived using the geodesic equation. The wavelength of a photon is inversely proportional to its momentum $\lambda = h/p$. As we show in Sec. A.3, the momentum of the photon is inversely proportional to the scale factor a and thus the wavelength scales as $a(t)$.

This approximation breaks down when the next-order term becomes of similar order of magnitude as $H_0 d$ or

$$\frac{\ddot{a}}{2a} d \gtrsim H_0 \qquad \Rightarrow \qquad d \gtrsim \frac{2H_0}{\frac{\ddot{a}}{a}} \tag{1.23}$$

In an era dominated by a cosmological constant (dark energy) with $\rho = -\mathcal{P} = \Omega_\Lambda \rho_{cr} = \Omega_\Lambda 3 H_0^2 m_P^2$ this condition becomes

$$d \gtrsim \frac{2\sqrt{\frac{\rho_\Lambda}{3m_P^2}}}{\frac{\rho_\Lambda}{3m_P^2}} = \frac{2\sqrt{3}m_P}{\sqrt{\Omega_\Lambda \rho_{cr}}} = \frac{2}{\sqrt{\Omega_\Lambda} H_0} \tag{1.24}$$

Thus the approximation breaks down for redshifts

$$z \gtrsim \frac{2}{\sqrt{\Omega_\Lambda}} \approx 2.4 \tag{1.25}$$

Note however that matter-dark energy equality is at $z \simeq 0.7$ and thus the above estimate is flawed and requires more careful treatment beyond $z \simeq 0.7$. Similarly, for small velocities $v \ll c$, the standard redshift formula can be used and we obtain $z \simeq v/c$ and thus the (recessional) velocity v can be expressed in terms of

$$v = H_0 D \tag{1.26}$$

with the proper distance $D = c(t_1 - t_0)$. This is denoted *Hubble's law* (See Fig. 1). It is thus a direct measure of the velocity of the galaxies.

1.3 Distances

There are two ways to measure distance, the comoving distance, χ, which remains fixed during expansion, and the physical distance, $d = a\chi$, which takes the expansion into account. As we are in an expanding space-time, we might wonder what is the more interesting physical distance: the distance at the time when the light was emitted or the distance when it was received. The well-defined measure of distance is a comoving distance. On a comoving grid, the distance simply amounts to $\left(dx^2 + dy^2 + dz^2\right)^{1/2}$. See Fig. 2 for an illustration.

1.3.1 Metric and comoving distance

Recall the form of the metric in terms of χ

$$ds^2 = a(t)^2 \left[-d\tau^2 + d\chi^2 + S_k^2(\chi) d\Omega^2\right] \tag{1.14}$$

where $S_k(\chi)$ is defined in Eq. (1.15). As light travels along null geodesics $ds^2 = 0$, the change in conformal time $\Delta\tau$ equals the change in comoving distance $\Delta\chi$, i.e. $\Delta\tau = \Delta\chi$ and thus the *comoving distance* between an object at time $t(a)$ and us is given by

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} \ . \tag{1.27}$$

Note that neither the comoving distance χ nor the metric distance d_m are observable.

1.3.2 Angular diameter distance

A common method to determine the distance of an object of known size D is to measure the angle $\delta\theta$ it takes on the sky as illustrated in Fig. 3a. Assuming that an object of known size D emits photons at time t_1 at comoving distance χ, then the *angular diameter distance* of the object is given by

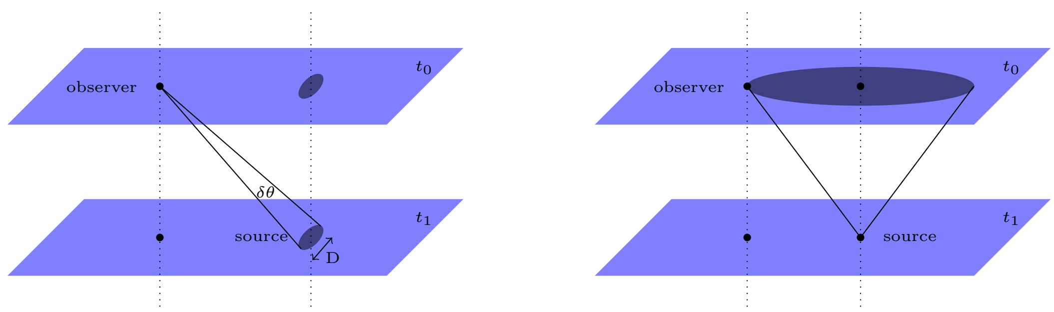
$$d_A \equiv \frac{D}{\delta\theta} \tag{1.28}$$

with $\delta\theta \ll 1$. In an FRW universe, the physical transverse size is given by

$$D = a(t_1) S_k(\chi) \delta\theta \tag{1.29}$$

and thus

$$d_A = \frac{d_m}{1+z} \; . \tag{1.30}$$



(a) Angular diameter distance

(b) Luminosity distance

Figure 3: Observable distances

1.3.3 Luminosity Distance

Another common way to infer the distance to an object of known luminosity is to compare the known to the measured luminosity L, i.e. energy emitted per second. One example are Type IA supernovae whose absolute luminosity is believed to be well understood. The observed flux F, energy per second per area, can be used to determine the *luminosity distance* d_L. The observed flux F at the *luminosity distance* d_L is given by

$$F = \frac{L}{4\pi d_L^2} \; . \tag{1.31}$$

Neglecting the expanding spacetime and any curvature, the luminosity distance would be simply given by the comoving distance χ. In an FRW spacetime the luminosity distance d_L the comoving distance χ has to be replaced by the metric distance d_m to account for possible curvature. Furthermore, at early times, the photons travel further on the comoving grid compared to today. Thus the number of photons received today is reduced by a factor $a(t_1) = 1/(1+z)$. Finally, if the photons are redshifted and thus the energy of the received photons is reduced by a factor a. Hence the measured flux using a general FRW metric is given by

$$F = \frac{L}{4\pi d_m^2 (1+z)^2} \; . \tag{1.32}$$

and a comparison with Eq. (1.31) shows that the luminosity distance is

$$d_L \equiv d_m(1+z) \ . \tag{1.33}$$

The luminosity distance is related to the angular diameter distance by

$$d_A = \frac{d_L}{(1+z)^2} \ . \tag{1.34}$$

The distance measurements were crucial to show that the universe is accelerating today, which lead to the award of the Nobel prize in physics for Saul Perlmutter, Brian Schmidt and Adam Riess in 2011. See Fig. 4

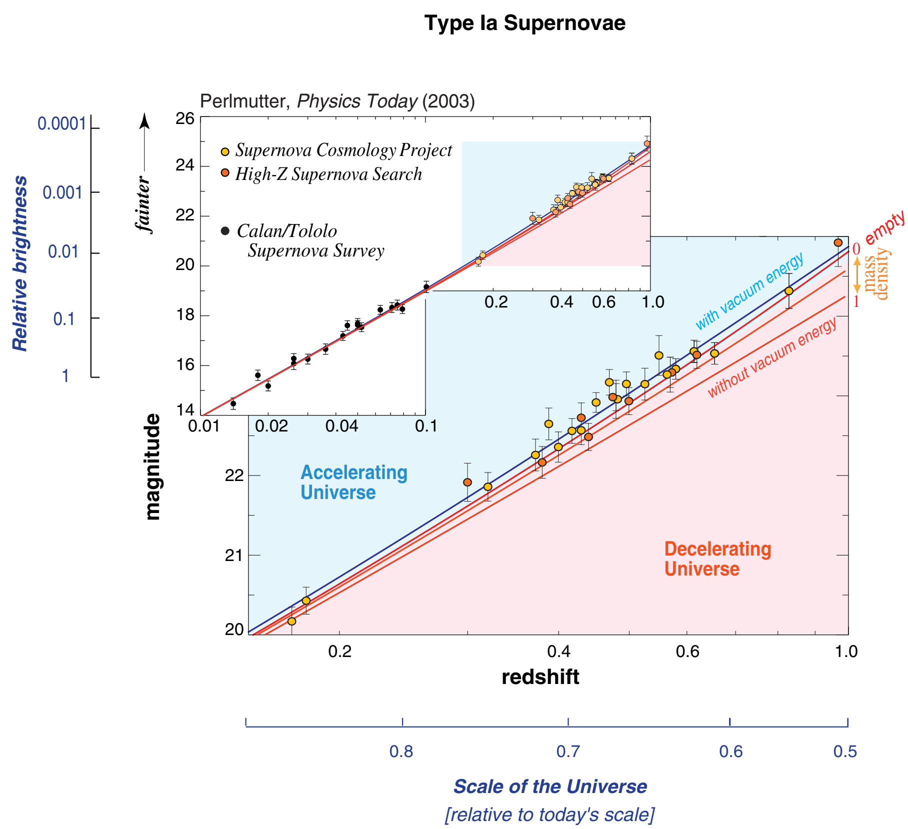


Figure 4: Hubble diagram [S. Perlmutter Physics Today April 2003 p.53]. Magnitude is the brightness (luminosity) on a logarithmic scale.

1.4 Horizons

The total comoving distance is given by the distance light could have travelled in a given time t, i.e.

$$\eta(t) = \int_0^t \frac{dt'}{a(t')} \ . \tag{1.35}$$

As nothing travels faster than light, $\eta(t)$ defines the *particle horizon*. We are not able to see anything in the past, which is beyond the *particle horizon*. It is monotonically increasing and can also be used as a measure of time and is the *conformal time* defined in Eq. (1.12). The proper distance of the particle horizon is given by

$$d_{max}(t) = a(t) \int_0^t \frac{dt'}{a(t')} \ . \tag{1.36}$$

Similarly, there might be a horizon for future events, if the universe recollapses at time T. Then the largest distance from which an observer might be able to receive signals travelling at the speed of light at any time later than t, is given by

$$\int_t^T \frac{dt'}{a(t')} \tag{1.37}$$

in comoving coordinates, which is denoted *event horizon*. The proper distance for an infinite distant future is given by

$$d_{MAX}(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} \ . \tag{1.38}$$

2 Cosmological Evolution

2.1 Friedmann Equations

We model the different components in the universe by perfect fluids as a leading approximation which are described by their energy density ρ and pressure $\mathcal{P}$. The evolution of the scale factor $a(t)$, energy density ρ and pressure $\mathcal{P}$ of the different fluids can be described by the *Friedmann equations*

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\mathcal{P}\right) \ , \tag{2.2}$$

which are the basic equations of FRW cosmology and follow directly from the Einstein equations. See App. B.1 and App. B.2 for a derivation in case of a flat Universe.

2.2 Continuity Equation

The expansion of the Universe can be considered as an adiabatic process (no heat transfer) and thus $dS = 0$ and the first law of thermodynamics is

$$dU = -\mathcal{P}dV \tag{2.3}$$

with the internal energy U, the pressure $\mathcal{P}$ and the volume V. The internal energy of the perfect fluid is given by $U = \rho V$ and the volume scales like $V = V_0 a^3 \propto a^3$. Thus we can rewrite the first law of thermodynamics as follows

$$\begin{aligned} 0 &= dU + \mathcal{P}dV = d(\rho V) + \mathcal{P}dV \qquad (2.4)\\ &= (\rho + \mathcal{P})\, dV + V d\rho \\ &= (\rho + \mathcal{P})\, V_0 da^3 + V_0 a^3 \dot{\rho} dt \\ &= (\rho + \mathcal{P})\, V_0 3a^2 \dot{a} dt + V_0 a^3 \dot{\rho} dt \end{aligned}$$

and we obtain the continuity equation[2]

$$\dot{\rho} + 3H\left(\rho + \mathcal{P}\right) = 0 \tag{2.5}$$

Introducing the *equation of state* and the *equation of state parameter* w

$$\mathcal{P} = w\rho \tag{2.6}$$

we can rewrite the continuity equation

$$0 = \frac{\partial \rho}{\partial t} + 3(1+w)\frac{\dot{a}}{a}\rho = a^{-3(1+w)}\frac{\partial\left(\rho a^{3(1+w)}\right)}{\partial t} \tag{2.7}$$

for constant equation of state parameter w and conclude $\rho \propto a^{-3(1+w)}$. We can insert this result into the Friedmann equation ($k = 0$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\,. \tag{2.8}$$

The solution gives us the time dependence of the scale factor for $w \neq -1$

$$a(t) \propto t^{\frac{2}{3(1+w)}} \qquad\qquad H = \frac{2}{3(1+w)t} \tag{2.9}$$

and for $\rho = \Lambda/8\pi G$ with $w = -1$, the scale factor is

$$a(t) \propto e^{\sqrt{\Lambda/3}t} \qquad\qquad H = \sqrt{\frac{\Lambda}{3}} \tag{2.10}$$

The second Friedmann equation determines whether the expansion is accelerating or decelerating

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\mathcal{P}\right) = -(1+3w)\frac{4\pi G}{3}\rho \tag{2.11}$$

and thus the expansion is accelerating for $w < -\frac{1}{3}$ and decelerating for $w > -\frac{1}{3}$.

2.3 Cosmic Inventory

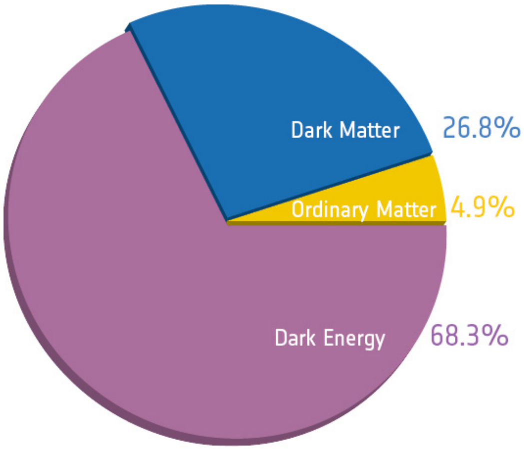
There are several different contributions to energy density of the Universe. Today, the most dominant contribution is dark energy (or a cosmological constant) and matter as shown in Fig. 5a, while radiation only contributes a small fraction. See Tab. 1 for a summary. The evolution of the three main components, dark energy, matter and radiation is shown in Fig. 5b.

2.3.1 Matter

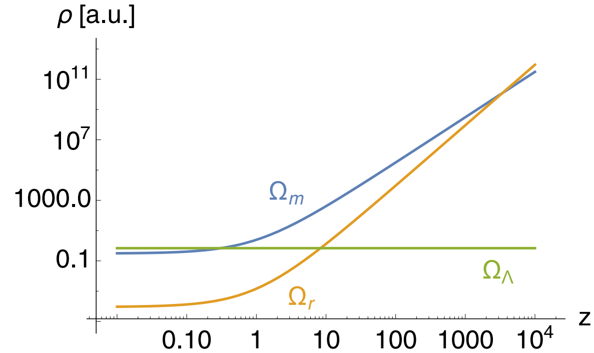
Matter refers to fluids with a negligible pressure, $|\mathcal{P}| \ll \rho$, which is a good description for a gas of non-relativistic particles. Thus setting $w = 0$, we obtain

$$\rho \propto a^{-3} \tag{2.12}$$

[2] The continuity equation is a direct result that the covariant divergence of the stress-energy tensor $\nabla_\mu T^{\mu\nu}$ vanishes. See App. B.3 for a discussion.



(a) Dominant contributions to the energy density of the Universe.



(b) Evolution of energy density for dark energy, matter and radiation.

Figure 5: Energy density composition today and its evolution.

	w	$\rho(a)$	$a(t)$	$H(t)$
matter	0	a^{-3}	$t^{2/3}$	$\frac{2}{3t}$
radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	$\frac{1}{2t}$
cosm.const.	-1	ρ_0	$e^{\sqrt{\Lambda/3}t}$	$\sqrt{\Lambda/3}$

Table 1: Evolution of different fluids

reflecting the expansion of the volume $V \propto a^3$. Ordinary matter (nuclei and electrons) are commonly called *baryonic matter*. Most of the matter in the Universe is in the form of dark matter, which is a new form of matter, which does not interact with photons, or at least extremely weakly. However its existence has been seen in numerous observations via its gravitational interactions at different length scales: The virial theorem $\left(\frac{1}{2}\left\langle v^2\right\rangle = \frac{GM}{R}\right)$ applied to COMA cluster (F.Zwicky 1933) shows the existence of additional non-baryonic matter, similarly galactic rotation curves [$\mathcal{O}(10s)$kpc], gravitational lensing [$< \mathcal{O}(200)$kpc], in a comparison of the observation of the bullet cluster in X-ray and gravitational lensing, the cosmic microwave background and large scale structure.

2.3.2 Radiation

The pressure of a relativistic gas of massless particles, for which the kinetic energy dominates the energy density, is one third of its energy density, $\mathcal{P} = \frac{1}{3}\rho$ and thus

$$\rho \propto a^{-4} \tag{2.13}$$

This can be easily understood by noticing that in addition to the decrease in the number density, the energy is redshifted $E \propto 1/a$.

The prime examples of radiation are

- *photons*, which we detect today as cosmic microwave background with temperature $T = 2.725$ K and small perturbations of order 10^{-5}.
- *neutrinos* in the early Universe. Today the masses of at least two neutrinos are relevant and thus they behave like matter.
- *gravitons*. Similar to photons and neutrinos there might be a background of gravitons (i.e. gravitational waves)

2.3.3 Dark energy

Matter and radiation are not enough to describe the evolution of the Universe. Recently dark energy became the dominant source of energy density in the Universe and constitutes about 70% of the total energy density today. The simplest explanation is in form of a cosmological constant with

$$\rho \propto a^0 \tag{2.14}$$

implying *negative* pressure $\mathcal{P} = -\rho$. A cosmological constant is predicted from quantum field theory. The ground state energy leads to a stress-energy tensor

$$T_{\mu\nu}^{vac} = \rho_{vac} g_{\mu\nu} \; . \tag{2.15}$$

However the *naive* prediction overestimates its size by several orders of magnitude $\rho_{vac}/\rho_{obs} \sim 10^{120}$, if a simple cutoff of the order of the Planck scale is imposed. Even though this naive estimate is questionable, the ground state energy is expected to change during phase transitions and thus it is an unsolved question why the cosmological constant is so small. Many alternatives to a cosmological constant have been suggested to address the smallness. Most explanations involve scalar fields and explain dark energy dynamically, but they generally do not address the problem of the vacuum energy.

2.4 Critical energy density

Sometimes it is convenient to express the energy density as a fraction of the critical energy density

$$\rho_{cr} = \frac{3H^2}{8\pi G} = 3H^2 m_P^2 \; , \tag{2.16}$$

where $m_P = (8\pi G)^{-1/2} \simeq 2 \times 10^{18}$ GeV is the reduced Planck mass. We define the fraction

$$\Omega_I = \frac{\rho_I}{\rho_{cr}} \tag{2.17}$$

and can rewrite the first Friedmann equation as

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \; . \tag{2.18}$$

As the energy densities $\Omega_r + \Omega_m + \Omega_\Lambda \approx 1$, we infer that $\Omega_k \approx 0$ and the Universe can be considered flat ($k = 0$). Alternatively we can express the first Friedmann equation in terms of the energy fractions today

$$\Omega_{I,0} = \frac{\rho_{I,0}}{\rho_{cr,0}} \tag{2.19}$$

with the critical energy density today

$$\rho_{cr,0} = 3H_0^2 m_P^2 \; . \tag{2.20}$$

The first Friedmann equation becomes

$$H^2(a) = H_0^2 \left[\Omega_{r,0} \left(\frac{a_0}{a}\right)^4 + \Omega_{m,0} \left(\frac{a_0}{a}\right)^3 + \Omega_{k,0} \left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0} \right] \tag{2.21}$$

with the Hubble parameter today H_0 and the scale factor a_0 today. In the following we will drop the subscripts '0' and use the normalisation $a_0 = 1$

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \ . \tag{2.22}$$

3 Thermal History

A basic understanding of the thermal history of the universe can be obtained by comparing the *rate of interactions* Γ to the rate of expansion H, or equivalently to its associated timescales $t_c \equiv \Gamma^{-1}$ and $t_H \equiv H^{-1}$. If the timescale for interactions is much smaller than the one for expansion

$$t_c \ll t_H \qquad \Leftrightarrow \Gamma \gg H \tag{3.1}$$

then *local thermal equilibrium* is reached. As the universe cools down, the interaction rate will decrease and when $t_c \sim t_H$, particles decouple from the thermal bath. Different particles decouple at different times. For example the interaction rate for $2 \to 2$ scattering is given by

$$\Gamma \equiv n\sigma v \tag{3.2}$$

with the number density n, the interaction cross section σ and the average velocity v. For ultra-relativistic particles ($v \sim 1$), the masses can be neglected and the only dimensionful quantity is the temperature with $n \sim T^3$ and $\sigma \sim \frac{\alpha^2}{T^2}$ for some exchange interaction with coupling constant α. All SM particles are ultra-relativistic for $T \gtrsim 100$GeV. Hence the interaction rate is

$$\Gamma = n\sigma v \sim T^3 \times \frac{\alpha^2}{T^2} = \alpha^2 T \tag{3.3}$$

while the Hubble rate scales like

$$H \sim \frac{\sqrt{\rho}}{m_P} \sim \frac{T^2}{m_P} \tag{3.4}$$

and thus the ratio

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 m_P}{T} \sim \frac{10^{14}\text{GeV}}{T} \tag{3.5}$$

and thus for $100\text{GeV} \lesssim T \lesssim 10^{14}$GeV particles are in local thermal equilibrium for $\alpha \sim 0.01$.

3.1 Equilibrium Thermodynamics

In local thermal equilibrium we can use distribution functions $f(\vec{x}, \vec{p})$, i.e. the occupation number of a small cell $d^3x d^3p/(2\pi\hbar)^3$ at position $(\vec{x}, \vec{p})$ to describe the fluids. In an homogeneous and isotropic universe, the distribution function does not depend on the position $\vec{x}$ and the direction of the momentum, but only the absolute magnitude of the momentum. The number density n_i of species i with g_i internal degrees of freedom is given by

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) \ . \tag{3.6}$$

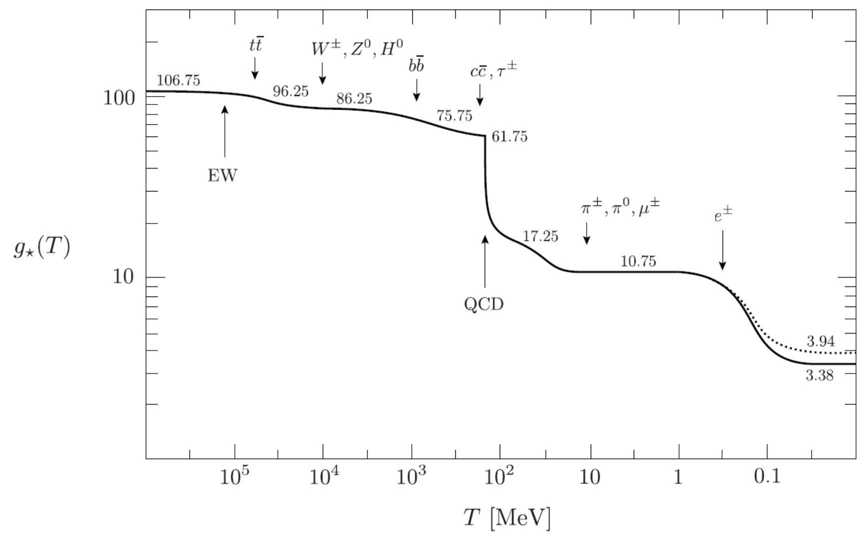


Figure 6: Evolution of effective relativistic degrees of freedom. Solid line is for $g_*^\rho(T)$ and the dotted line for $g_*^s(T)$. Taken from Cosmology lecture notes of Daniel Baumann.

Bosons and fermions follow the usual Bose-Einstein and Fermi-Dirac distributions in equilibrium at a temperature T respectively

$$f(p) = \frac{1}{e^{(E-\mu)/T} \pm 1} \tag{3.7}$$

with + for the Fermi-Dirac and − for the Bose-Einstein distribution. Similarly we can define the energy density and the pressure

$$\rho = g_i \int \frac{d^3p}{(2\pi)^3} f(p) E(p) \overset{T \gg m,\mu}{\longrightarrow} \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{bosons} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{fermions} \end{cases} \tag{3.8}$$

$$\mathcal{P} = g_i \int \frac{d^3p}{(2\pi)^3} f(p) \frac{p^2}{3E(p)} \overset{T \gg m,\mu}{\longrightarrow} \frac{1}{3}\rho \,. \tag{3.9}$$

See exercise 15 in chapter 2 of [1] to understand the form of the expressions for the energy density and the pressure. The solutions to the exercise are provided in the appendix of [1]. The entropy density is related to energy density and pressure as follows (App. B.4)

$$s(T) = \frac{\partial \mathcal{P}}{\partial T} \qquad s(T) = \frac{\rho(T) + \mathcal{P}(T)}{T} \,. \tag{3.10}$$

In the radiation dominated era, it is convenient to define the effective relativistic degrees of freedom $g_*^{\rho,s}(T)$ as follows

$$\rho = \frac{\pi^2}{30} g_*^\rho(T) T^4 \qquad\qquad s = \frac{2\pi^2}{45} g_*^s(T) T^3 \,. \tag{3.11}$$

Whenever particles decouple from the thermal plasma, $g_*^{s,\rho}$ decreases. For most of the time, $g_*^\rho(T) = g_*^s(T)$, as it is shown in Fig 6.

Real scalars have one degree of freedom, complex scalars two, massless gauge bosons two (polarizations), massive gauge bosons three (polarizations), Weyl fermions two, Majorana fermions two, and Dirac fermions four (left- and right-handed and (anti-)particles).

3.2 Relativistic Decoupling of Neutrinos

Neutrinos are almost massless fermions. They decouple from the cosmic plasma around 1 MeV and thus shortly before electrons and positrons become non-relativistic and reheat the cosmic plasma. Thus neutrinos are effectively colder than the cosmic plasma, since they are not reheated by electron-positron pair annihilation. Using entropy conservation, we find for the entropy before neutrino decoupling at scale factor a_1

$$s(a_1) = \frac{2\pi^2}{45} T_1^3 \left[2 + \frac{7}{8} (2 + 2 + 3 \cdot 2) \right] = \frac{43\pi^2}{90} T_1^3 \ , \tag{3.12}$$

because there are in total 2 degrees of freedom from the two polarisations of photons, 2 spin degrees of freedom for both electrons and positrons and 3 generations of neutrinos with spin 2. After electrons and positrons become non-relativistic, they transfer their entropy to the cosmic plasma and effectively reheat the cosmic plasma. Hence the entropy at a late-enough redshift a_2 is given by

$$s(a_2) = \frac{2\pi^2}{45} \left[2T_\gamma^3 + \frac{7}{8} 6T_\nu^3 \right] \ , \tag{3.13}$$

where photons have a temperature T_γ and neutrinos have temperature T_ν. Entropy conservation $s(a_1)a_1^3 = s(a_2)a_2^3$ results in

$$\frac{43}{2} (a_1 T_1)^3 = 4 \left[\left(\frac{T_\gamma}{T_\nu} \right)^3 + \frac{21}{8} \right] (T_\nu(a_2) a_2)^3 \ . \tag{3.14}$$

Finally we have to relate the temperature T_1 to the temperature at a later time. After neutrinos are decoupled, they still preserve the shape of the Fermi-Dirac distribution and the temperature is inversely proportional to the scale factor. This can be directly seen from observing that the energy of a massless particle scales like a^{-1} as shown in Eq. (A.28). Thus the temperature of neutrinos T_ν satisfies $a_2 T_\nu = a_1 T_1$. Solving Eq. (3.14) for the temperature of neutrinos T_ν, we obtain

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3} \tag{3.15}$$

and conclude that the temperature of neutrino background today is lower compared to the cosmic microwave background. We find for the temperature of neutrinos today

$$T_\nu^0 = T_\gamma^0 \left(\frac{4}{11} \right)^{1/3} = 2.73 \left(\frac{4}{11} \right)^{1/3} K = 1.95K = 1.68 \times 10^{-4} \text{eV} \ . \tag{3.16}$$

It has been undeniably shown that neutrinos are massive. The temperature of neutrinos today T_ν^0 is smaller than the square root of the solar mass squared difference, $\sqrt{\Delta m_\odot^2} = 8.66 \times 10^{-3}$eV, and thus at least two neutrinos are non-relativistic today and their mass can not be neglected. The energy density of one neutrino is given by

$$\rho_{1\nu} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_\nu^2}}{e^{p/T_\nu} + 1} \tag{3.17}$$

and shown in Fig. 7. Thus the total energy density in neutrinos is dominated by their mass $\rho_\nu = m_\nu n_\nu$

$$\Omega_\nu h^2 = \frac{m_\nu}{94\text{eV}} \ . \tag{3.18}$$

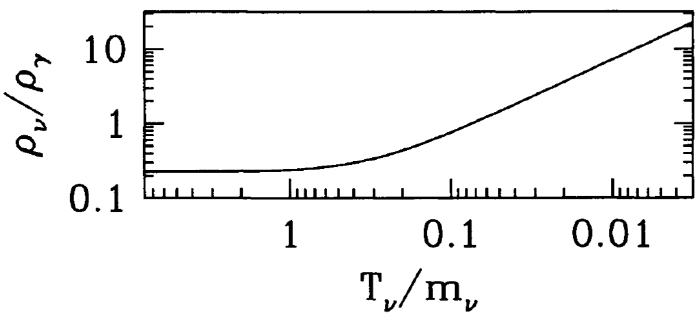


Figure 7: Neutrino energy compared to photon energy vs temperature of neutrinos. Taken from Dodelson[1]

4 The Boltzmann Equation for the Number Density

The Boltzmann equation for the number density n_1 of the first particle in case of two to two scatterings, $1 + 2 \leftrightarrow 3 + 4$ is given by

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = \int \prod_{i=1}^{4} d\Pi_i \,(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,|\mathcal{M}|^2 \\ \times \left\{ f_3 f_4 \left(1 \pm f_1\right)\left(1 \pm f_2\right) - f_1 f_2 \left(1 \pm f_3\right)\left(1 \pm f_4\right) \right\} \quad (4.1)$$

with the phase space integrals

$$\int d\Pi_i = g_i \int \frac{d^3 p_i}{(2\pi)^3 2E_i(p_i)} \quad (4.2)$$

for particle i with g_i internal degrees of freedom. See App. C for a derivation from the Boltzmann equation for the number density. The phase space integral is Lorentz invariant

$$\int \frac{d^3 p}{2E(p)}\delta(E - \sqrt{p^2 + m^2}) = \int d^3 p \int dE \delta(E^2 - p^2 - m^2)\theta(E) \quad (4.3)$$

because we implicitly impose that the particle are on-shell, i.e. satisfy the energy-momentum dispersion relation

$$E^2 = p^2 + m^2 \quad (4.4)$$

Let us understand the different factors in the Boltzmann equation. In the absence of any interactions, the right-hand side of the equation, the Boltzmann equation tells us that the number of particles in a comoving volume does not change. However the number of particles in a physical volume scales like a^{-3} due to the expansion. Interactions between the different particles are described by the integral on the right-hand side. The integrals are over the whole phase space $\int d\Pi_i$ of the different particles involved in the interaction. Energy-momentum conservation is imposed by the four-dimensional delta function. The factor $|\mathcal{M}|^2$ is the square of the amplitude (matrix element), which governs the strength of the interaction. For example in the case of Compton scattering it is proportional to the fine-structure constant α^2. It is averaged over *initial and final states*. The last factor on the right-hand side consists of two terms and takes into account the occupation numbers (distribution functions) of the different states. The first term is proportional to $f_3 f_4(1 \pm f_1)(1 \pm f_2)$ and describes the production of a particle 1 in the process $3 + 4 \to 1 + 2$, i.e. it is proportional to the initial abundances and the factors $(1 \pm f_i)$ take into account the possible Pauli-blocking for fermions with a minus sign or Bose enhancement for bosons with a plus sign. The second term describes the destruction of particle 1 in the process $1+2 \to 3+4$. The first term is sometimes called *source term* and the second *loss term*. Note that we assumed that the process is reversible.

Usually scattering between the different particles enforces kinetic equilibrium, i.e. the different particle species follow the Bose-Einstein or Fermi-Dirac statistics, however they are not necessarily in chemical equilibrium. If they were the chemical potential μ would balance against the other chemical potentials. In the case of $e^+ + e^- \leftrightarrow \gamma\gamma$, we would find $\mu_{e^+} + \mu_{e^-} = 2\mu_\gamma$.

For systems at temperature $T \ll E - \mu$ we can neglect the terms ± 1 in the denominators of the Fermi-Dirac and Bose-Einstein distributions and work with the Maxwell-Boltzmann distribution

$$f(E) = e^{-(E-\mu)/T} = e^{\mu/T} e^{-E/T} \,. \tag{4.5}$$

Similarly we can neglect the Pauli-blocking/Bose enhancement factors and can approximate

$$\begin{aligned} &\{f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)\} \\ &\qquad \to f_3^{MB} f_4^{MB} - f_1^{MB} f_2^{MB} = e^{-(E_1+E_2)/T} \left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right) \end{aligned} \tag{4.6}$$

using energy-momentum conservation. The number density

$$n_i = n_i^{(0)} e^{\mu_i/T} \tag{4.7}$$

of species i can be expressed as a function of μ_i and the *equilibrium number density*

$$n_i^{(0)} = g_i \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} & m_i \gg T \\ g_i \zeta(3) \frac{T^3}{\pi^2} & m_i \ll T \end{cases} . \tag{4.8}$$

Using this we can rewrite Eq. (4.6)

$$e^{-(E_1+E_2)/T} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \tag{4.9}$$

and consequently the Boltzmann equation

$$a^{-3} \frac{d\left(n_1 a^3\right)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \tag{4.10}$$

where we defined the thermally averaged cross section

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int d\Pi_i e^{-(E_1+E_2)/T} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \left|\mathcal{M}\right|^2 \,. \tag{4.11}$$

Before moving on, a few comments are in order. Note that we could equally well use $E_3 + E_4$ and the equilibrium number densities of the particles 3 and 4. It is straightforward to generalise the expression to other processes, like decays $(1 \to 2)$ processes or scattering with more than 2 particles in the final state.

The Boltzmann equation can be applied to many processes in the early Universe. We will discuss big bang nucleosynthesis (BBN) and the freeze-out of a massive particle, which is relevant for dark matter production, in detail and defer the study of recombination to the assignment.

If the interaction rate $\langle\sigma v\rangle n_2^{(0)}$ is large compared to the Hubble rate, the Boltzmann equation (4.10) can only be satisfied if the number densities satisfy

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{4.12}$$

and consequently the chemical potential are related by

$$\mu_3 + \mu_4 = \mu_1 + \mu_2 \; . \tag{4.13}$$

This case is commonly denoted *chemical equilibrium* in the context of the production of heavy relics, *nuclear statistical equilibrium* in the context of big bang nucleosynthesis, and *Saha equation* in the context of recombination.

4.1 Freeze-Out

The prime example for freeze-out is dark matter production via freeze-out. We consider a massive Dirac particle X with mass m_X, which is initially in thermal equilibrium with the cosmic plasma, but later freezes-out, i.e. decouples from the thermal SM plasma. Let us consider processes of the type $X\bar{X} \leftrightarrow l\bar{l}$, where the pair of particles $X\bar{X}$ annihilate into a pair of light particles $l\bar{l}$ and vice versa. We assume that the light particle is in chemical as well as kinetic equilibrium with the cosmic plasma, i.e. $n_l = n_l^{(0)}$. Thus we find for the Boltzmann equation of $n_X = n_{\bar{X}}$ (4.10)

$$a^{-3}\frac{d\left(n_X a^3\right)}{dt} = \langle\sigma v\rangle \left\{ \left(n_X^{(0)}\right)^2 - n_X^2 \right\} \; . \tag{4.14}$$

We will assume that g_* is constant, which is a good approximation for temperature well above the QCD phase transition. We define

$$Y \equiv \frac{n_X}{s} \qquad \text{and} \qquad Y_{(0)} \equiv \frac{n_X^{(0)}}{s} \tag{4.15}$$

to rewrite the differential equation for the number density in the convenient form

$$\frac{dY}{dt} = s\,\langle\sigma v\rangle \left(Y_{(0)}^2 - Y^2\right) \; . \tag{4.16}$$

The freeze-out process is characterised by the mass m_X of the particle X. Thus it is convenient to express the temperature in terms of m_X as follows

$$x = \frac{m_X}{T} \; . \tag{4.17}$$

In the radiation dominated era, the first Friedmann equation can be written as

$$H(T) = \left(\frac{1}{2t}\right) = \sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30} g_*^\rho(T) T^2} = \sqrt{\frac{8\pi^3 G}{90} g_*^\rho(T)}\frac{m_X^2}{x^2} = \frac{H(m_X)}{x^2} \tag{4.18}$$

and consequently the evolution equation can be rewritten as (assuming that g_*^ρ is constant)

$$\frac{x}{Y_{(0)}}\frac{dY}{dx} = -\frac{\Gamma}{H}\left(\frac{Y^2}{Y_{(0)}^2} - 1\right) \tag{4.19}$$

with the interaction rate $\Gamma = n_X^{(0)} \langle\sigma v\rangle$.

There is no general analytic solution. However we can obtain an approximate analytic solution. While $\Gamma \gg H$, the abundance Y of the particle X will track its equilibrium value $Y_{(0)}$

$$Y_0(x) = \begin{cases} \frac{45}{2\pi^4}\sqrt{\frac{\pi}{8}}\frac{g}{g_*^s}x^{3/2}e^{-x} & x \gg 3 \\ \frac{45\zeta(3)}{2\pi^4}\frac{g}{g_*^s} & x \ll 3 \text{ (bosons)} \\ \frac{45\zeta(3)}{2\pi^4}\frac{3}{4}\frac{g}{g_*^s} & x \ll 3 \text{ (fermions)} \end{cases} \tag{4.20}$$

Note that the average momentum $\langle p\rangle/n \sim 3T$ for Fermi-Dirac, Bose-Einstein and Maxwell-Boltzmann distributions. When $\Gamma \simeq H$, the particle decouples from the thermal plasma. It freezes out. We denote the freeze-out temperature by T_f or $x_f = m_X/T_f$. If the particle freezes out while still being relativistic. Its final abundance is given by its equilibrium value at freeze-out

$$Y_\infty = Y_{(0)}(x_f) \ . \tag{4.21}$$

Assuming the expansion is isentropic (constant entropy per comoving volume), its abundance today is $n_X^0 = s_0 Y_\infty$. If it later becomes non-relativistic, its energy density is given by $\rho_X^0 = s_0 Y_\infty m_X$. For example for a single 2-component neutrino species we find

$$\Omega_\nu h^2 = \frac{s_0 m_\nu}{\rho_{cr}}\frac{45\zeta(3)}{2\pi^4}\frac{3}{4}\frac{2}{g_*^s}h^2 \simeq \frac{m_\nu}{94\text{eV}} \tag{4.22}$$

If the particle however is already non-relativistic when decoupling, at late times for $T \ll m_X$, i.e. $x \gg 1$, we can obtain an approximate solution for late times when the equilibrium abundance is exponentially suppressed. We parameterize the interaction rate by (A prime denotes a derivative with respect to x)

$$Y' = -\lambda x^{-n-2}(Y^2 - Y_{(0)}^2) \qquad\qquad \lambda = \left[\frac{x\langle\sigma v\rangle s}{H(m_X)}\right]_{x=1} \tag{4.23}$$

where we parameterize the temperature dependence of the cross section by n:

$$\langle\sigma v\rangle = \langle\sigma v\rangle_{x=1} x^{-n} \ . \tag{4.24}$$

The temperature dependence of the cross section comes from its velocity dependence $\sigma v \propto v^p$, where $p = 0$ corresponds to s-wave annihilation, $p = 2$ to p-wave annihilation and so on. Since $\langle v\rangle \sim T^{1/2}$, we find $\langle\sigma v\rangle \propto T^{p/2}$. We introduce $\Delta = Y - Y_{(0)}$ and thus the Boltzmann equation becomes

$$\Delta' = -Y_{(0)}' - \lambda x^{-n-2}\Delta(2Y_{(0)} + \Delta) \tag{4.25}$$

For $1 \lesssim x \ll x_f$, $Y \sim Y_{(0)}$ and thus Δ, Δ' are small. In fact $\Delta' \ll \Delta$. We obtain the approximate solution using $Y_{(0)}' \simeq -Y_{(0)}$ (The approximation for the derivative of the equilibrium number density holds for $x \gg 1$.)

$$\Delta \simeq \frac{x^{n+2}}{2\lambda} \tag{4.26}$$

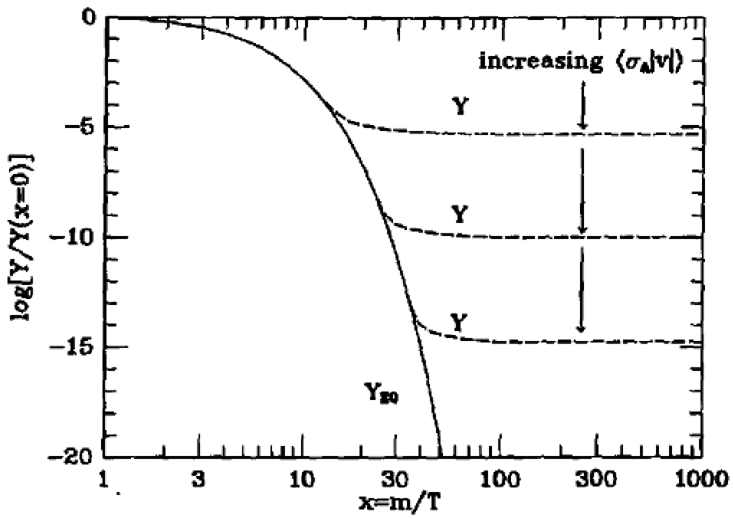


Figure 8: Taken from Kolb/Turner[3].

and for $x \gg x_f$, we expect $\Delta \simeq Y \gg Y_{(0)}$ and thus

$$\Delta' = -\lambda x^{-n-2}\Delta^2 \tag{4.27}$$

which can be integrated from $x = x_f$ to infinity to obtain

$$Y_\infty = \Delta_\infty = \frac{n+1}{\lambda} x_f^{n+1} \tag{4.28}$$

We finally have to determine the freeze-out temperature. A simple criterion is given by $H(x_f) \simeq \Gamma(x_f)$, which

$$H(m_X) = x_f^2 \Gamma(x_f) = x_f^2 \langle\sigma v\rangle n^{(0)}(x_f) \simeq \frac{g_X m_X^3 \langle\sigma v\rangle}{(2\pi)^{3/2}} x_f^{1/2} e^{-x_f} \ . \tag{4.29}$$

See section 5.2 in Kolb/Tuerner for a more refined determination. Typical values for x_f are a few times 10. See Fig. 8. The final number density is then simply given by

$$n_{X,\infty} = s_0 Y_\infty \tag{4.30}$$

of in terms of the energy density

$$\Omega_X = \frac{n+1}{\lambda} x_f^{n+1} \frac{m_X s_0}{\rho_{cr}} \approx 0.2 \frac{x_f}{20} \sqrt{\frac{g_*^\rho(m_X)}{100}} \frac{3 \times 10^{-26} \mathrm{cm}^3/\mathrm{sec}}{\langle\sigma v\rangle} \ . \tag{4.31}$$

This is a remarkable result, which nicely ties in with particle physics, because the cross section needed to obtain the correct relic abundance for a particle X with masses of ~ 100 GeV is of order of the weak-interaction cross section G_F^2. This coincidence is often called *WIMP miracle*, because a weakly interacting massive particle (WIMP) automatically obtains the correct abundance via freeze-out to explain dark matter. They naturally appear in many theories beyond the Standard Model (SM) of particle physics, like the lightest supersymmetric particle (LSP) in the minimal supersymmetric SM. There is a big experimental effort to search for these particles using all possible means: colliders, direct and indirect detection experiments. All three possible channels are related via crossing symmetry with the cross section relevant for dark matter pair annihilation in the early Universe, as it is shown in Fig. 9. WIMPs are particularly constrained by direct detection searches as shown in Fig. 10.

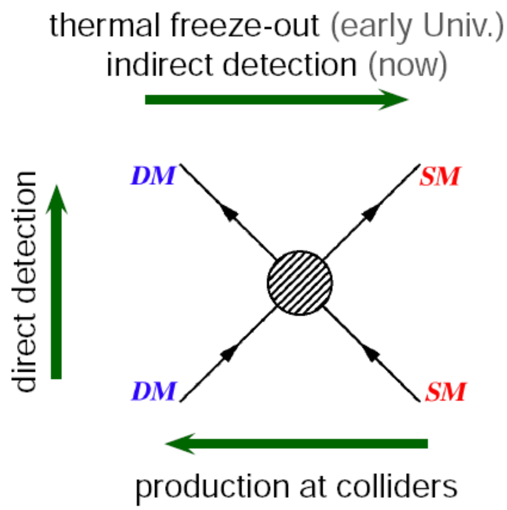


Figure 9: Crossing symmetry. Taken from `http://www.mpi-hd.mpg.de/lin`.

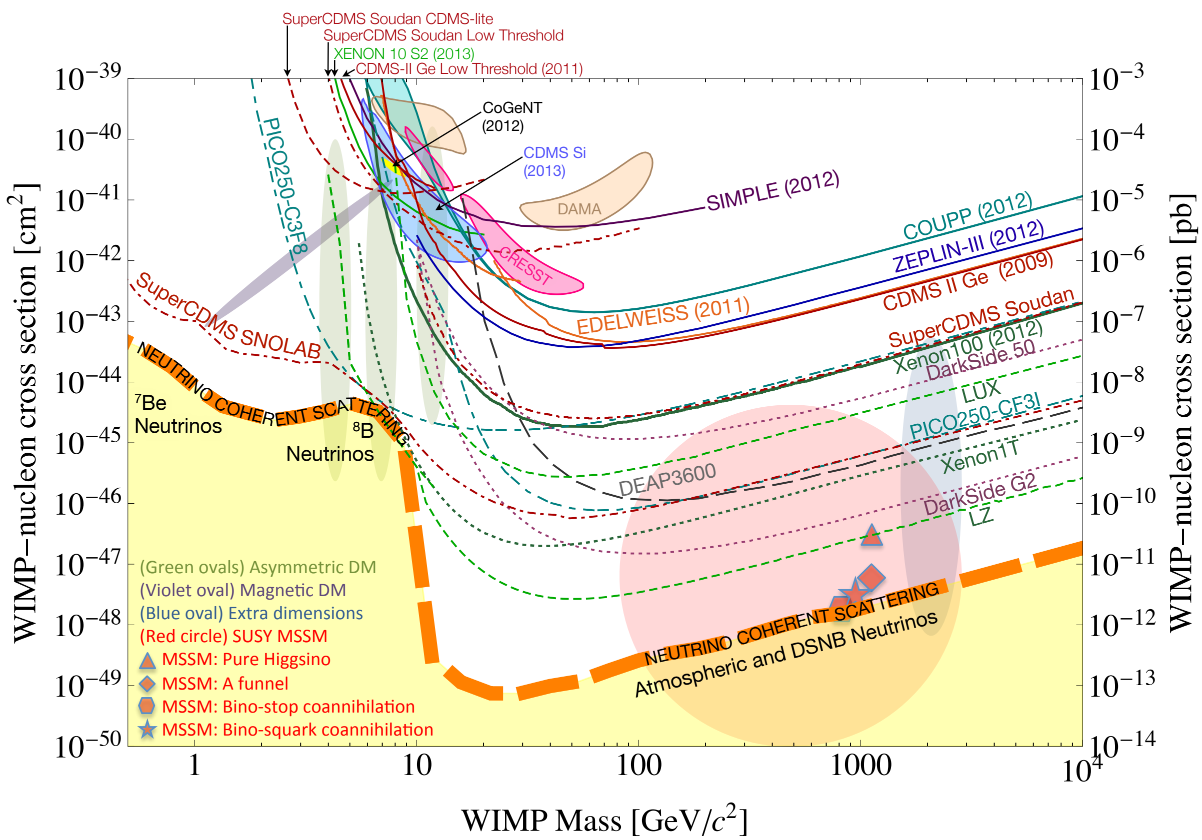


Figure 10: Dark matter direct detection.[6]

A Spacetime

A.1 Metric

In order to understand the Hubble diagram, we have to learn how to measure distances and length scales in the Universe. Before looking at distances in space-time, let us first consider distances in Euclidean space. In Euclidean space, the distance between two points is given by the distance in x and y direction between the two points in Cartesian coordinates

$$ds^2 = dx^2 + dy^2 \tag{A.1}$$

where we used Cartesian coordinates to write the distance in the last term. However the result should not depend on the chosen coordinate system. Thus choosing polar coordinates ($r = \sqrt{x^2 + y^2}$, θ) with

$$x = r \sin\theta \qquad\qquad y = r\cos\theta \ , \tag{A.2}$$

we find for a distance between two points

$$ds^2 = dr^2 + r^2 d\theta^2 \ . \tag{A.3}$$

In general we can write

$$ds^2 = \sum_{ij} g_{ij}(x) dx^i dx^j \ , \tag{A.4}$$

where g is a symmetric matrix, which is called *metric*. The metric defines a scalar product on the vector space and consequently a norm, which can be used to define distances. In four space-time dimensions, we conventionally write

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}(x) dx^\mu dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu \ . \tag{A.5}$$

The $\sum$ sign is often dropped and it is convention to sum over the same index, if it appears as lower and upper index. ds^2 is sometimes called *proper time*. The metric g has 10 degrees of freedom.

One special case is special relativity with the metric

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \ . \tag{A.6}$$

The *signature* of the metric is the number of eigenvalues ± 1 of the metric. In case of special and general relativity it is $(3,1)$ or $(1,3)$ depending on the convention whether the time component has eigenvalue ± 1. We will follow the convention in Dodelson [1]. Note, the lecture notes by Daniel Baumann [2] use the signature $(1,3)$.

A.2 Geodesics

How does a particle move without any external forces? Newton's law tells us

$$\frac{d^2 x^i}{dt^2} = 0 \ . \tag{A.7}$$

How can we generalise this to a general coordinate system? For example for a system in polar coordinates, $x' = (r, \theta)$, the equations of motion look different. Starting from a Cartesian coordinate system, we find

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt} \,. \tag{A.8}$$

with the *transformation matrix* $\partial x^i/\partial x'^j$. In case of polar coordinates

$$x^1 = x'^1 \cos x'^2 \qquad\qquad x^2 = x'^1 \sin x'^2 \tag{A.9}$$

the transformation matrix is

$$\frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \cos x'^2 & -x'^1 \sin x'^2 \\ \sin x'^2 & x'^1 \cos x'^2 \end{pmatrix} \,. \tag{A.10}$$

Applying the second derivative and doing the algebra we find

$$0 = \frac{d^2x^i}{dt^2} = \frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = \frac{\partial x^i}{\partial x'^j}\frac{d^2x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} \tag{A.11}$$

multiplying with the inverse of the transformation matrix we obtain

$$\frac{d^2x'^l}{dt^2} + \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^l_{\;i} \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0 \,. \tag{A.12}$$

Solutions to the this equation are called geodesics and the equation itself is commonly denoted geodesic equation. There are two small changes in general relativity, the index runs from 0 to 3 and we can not use time t to parameterize the path, but we have to use different monotonically increasing parameter along the geodesic. With these modifications we can rewrite the geodesic equation as

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{A.13}$$

with the Christoffel symbol

$$\Gamma^\mu_{\alpha\beta} = \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^\mu_{\;\kappa} \frac{\partial^2 x^\kappa}{\partial x'^\alpha \partial x'^\beta} \tag{A.14}$$

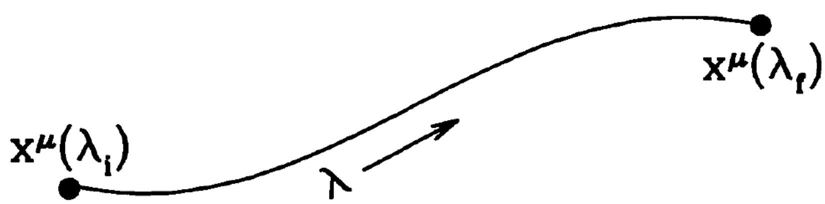


Figure 11: Curve in space-time. Copied from [1]

In the absence of any (non-gravitational) forces, particles move along *geodesics*, the curve of least action. This path is determined by the *geodesic equation*

$$\frac{dU^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta \tag{A.15}$$

for a particle with mass m and velocity

$$U^\mu = \frac{dx^\mu}{d\lambda} \tag{A.16}$$

where λ can be in principle any parameter parameterising the curve. One convenient choice is the proper time of the particle. In the following we will use $\lambda = \tau$. The *Christoffel symbol* (for a metriccompatible connection) is defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] . \tag{A.17}$$

In the general relativity course you will learn that you can rewrite the geodesic equation in a more compact form using the *covariant derivative* ∇_α

$$U^{\alpha}\left(\frac{\partial U^{\mu}}{\partial X^{\alpha}} + \Gamma^{\mu}_{\alpha\beta}U^{\beta}\right) \equiv U^{\alpha}\nabla_{\alpha}U^{\mu} = 0 \tag{A.18}$$

where we made use of the chain rule

$$\frac{d}{d\tau}U^{\mu}(X^{\alpha}(\tau)) = \frac{dX^{\alpha}}{d\tau}\frac{\partial U^{\mu}}{\partial X^{\alpha}} = U^{\alpha}\frac{\partial U^{\mu}}{\partial X^{\alpha}} . \tag{A.19}$$

In terms of the 4-momentum, $P^{\mu} \equiv mU^{\mu} = (E, \vec{P})$, the geodesic equation becomes

$$P^{\alpha}\frac{\partial P^{\mu}}{\partial X^{\alpha}} = -\Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta} \tag{A.20}$$

which is also valid for a massless particle.

A.3 Point Particle in FRW Universe

The Christoffel symbol for a flat FRW metric has only a few non-vanishing components

$$\Gamma^{0}_{ij} = \dot{a}a\gamma_{ij} \qquad\qquad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \frac{\dot{a}}{a}\delta^{i}_{j} \tag{A.21}$$

$$\Gamma^{i}_{jk} = 0 \quad \text{for Euclidean space } (k = 0) \tag{A.22}$$

which can be easily derived using Eqs. (A.17) and (1.6) .

In an FRW universe, we have $\partial_i P^{\mu} = 0$ due to homogeneity and thus the geodesic equation reduces to

$$P^{0}\frac{dP^{\mu}}{dt} = -\Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta} = -\left(2\Gamma^{\mu}_{0j}P^{0} + \Gamma^{\mu}_{ij}P^{i}\right)P^{j} \tag{A.23}$$

Thus we immediately see that particles at rest remain at rest

$$P^{j} = 0 \Rightarrow \frac{dP^{j}}{dt} = 0 . \tag{A.24}$$

Considering the zeroth component

$$E\frac{dE}{dt} = -\Gamma^{0}_{ij}P^{i}P^{j} = -\frac{\dot{a}}{a}p^{2} \tag{A.25}$$

using $E = P^0$ and the physical 3-momentum $p^2 = a^2\gamma_{ij}P^iP^j$.

Using the on-shell condition of the particle

$$-m^{2} = g_{\mu\nu}P^{\mu}P^{\nu} = -E^{2} + p^{2} \tag{A.26}$$

and thus $EdE = pdp$, the geodesic equation implies

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a} \Rightarrow p(t) \propto \frac{1}{a(t)} . \tag{A.27}$$

Hence physical 3-momentum "decays". In particular for massless particles energy decays and physical peculiar velocity $v^2 = a^2\gamma_{ij}v^iv^j$ decay,

$$E = p \propto \frac{1}{a(t)} \qquad \text{(massless particles)} \tag{A.28}$$

$$p = \frac{mv}{\sqrt{1-v^2}} \propto \frac{1}{a} \qquad \text{(massive particles)}\ , \tag{A.29}$$

with the comoving peculiar velocity $v^i = dX^i/dt$ (the velocity relative to the comoving frame). The comoving peculiar velocity is related to the comoving momentum

$$P^i = mU^i = m\frac{dX^i}{d\tau} = mv^i\frac{dt}{d\tau} = \frac{mv^i}{\sqrt{1-a^2\gamma_{ij}v^iv^j}} = \frac{mv^i}{\sqrt{1-v^2}} \tag{A.30}$$

with $v^2 \equiv a^2\gamma_{ij}v^iv^j$. The next-to-last equality follows from the relation of the τ on t can be obtained from the metric

$$d\tau^2 = -ds^2 = dt^2 - a^2\gamma_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt^2 = dt^2 - a^2\gamma_{ij}v^iv^jdt^2 = (1-v^2)dt^2 \tag{A.31}$$

Thus free-falling particles will asymptotically approach the Hubble flow.

B Derivation of Friedmann Equations

B.1 Einstein Equation

The metric introduced in the previous sections describes gravity and the interaction of gravity with matter is described by the Einstein equation[3]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{B.2}$$

with Newton's constant G, the Einstein tensor $G_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $\mathcal{R}$, and the energy-momentum tensor $T_{\mu\nu}$. The Ricci tensor and Ricci scalar describe the curvature of space-time. The Ricci scalar is simply defined by the contraction of the Ricci tensor with the metric

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} \tag{B.3}$$

and the Ricci tensor can be obtained from the Christoffel symbols[4]

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}\ . \tag{B.4}$$

[3]A possible cosmological constant Λ is absorbed in the energy-momentum tensor, i.e.

$$T^\Lambda_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu}\ . \tag{B.1}$$

[4]The curvature is defined similar to the field strength tensor in quantum field theory from the commutator of the covariant derivatives $[\nabla_\mu, \nabla_\nu]$. Please refer to a general relativity book.

In particular looking at the 00 component of the Ricci tensor in an FRW metric we find

$$R_{00} = \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \tag{B.5}$$

$$= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} \tag{B.6}$$

$$= -\frac{\partial}{\partial t}\delta^i_i \frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \delta^i_j \delta^j_i = -3\frac{\ddot{a}}{a} \; . \tag{B.7}$$

Similarly for the spatial components ($k = 0$) we find

$$R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \tag{B.8}$$

and the Ricci scalar can be evaluated to

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = -R_{00} + \frac{1}{a^2} R_{ii} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) \; . \tag{B.9}$$

Plugging everything into Einstein equations we obtain two independent equations describing the evolution in a flat ($k = 0$) FRW Universe

$$R_{00} - \frac{1}{2} g_{00} \mathcal{R} = 3\left(\frac{\dot{a}}{a}\right)^2 = 3H^2 = 8\pi G T_{00} \tag{B.10}$$

$$g^{\mu\nu} G_{\mu\nu} = -\mathcal{R} = 8\pi G T^\mu_\mu$$

B.2 Perfect Fluid

Before interpreting the Einstein equations, we have to have a closer look at the energy-momentum tensor $T^{\mu\nu}$. Our basic assumption is that we can describe the content of the Universe by different perfect fluids as a leading approximation, i.e. the fluid can be described by macroscopic quantities, its energy density and pressure, while there is no stress or viscosity in agreement that with the metric being homogeneous and isotropic.

The energy momentum tensor describes the flux of four-momentum p^μ in the direction x^ν. The energy-momentum tensor of a perfect fluid in its rest-frame in Minkowski space is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \tag{B.11}$$

Due to isotropy it is diagonal and its spatial components have to be equal. The 00-element is just the energy density ρ, i.e. the flux of energy density in time direction, while the spatial elements ii are given by the flux of momentum density p_i in the direction x_i, i.e. the pressure $\mathcal{P}_i = \frac{dp_i}{dt} dx_i$ in direction x_i. In order to write it in a covariant form, we first introduce the four-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \tag{B.12}$$

with the *proper time*

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \; . \tag{B.13}$$

For a particle at rest we find $U^\mu = (1, 0, 0, 0)$. Hence we can write the energy-momentum tensor as

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P}\eta^{\mu\nu} \tag{B.14}$$

and its generalisation to general relativity is straightforward

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P} g^{\mu\nu} \; . \tag{B.15}$$

Thus we find in the rest-frame of the fluid in the FRW universe

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}\mathcal{P} & & \\ & & a^{-2}\mathcal{P} & \\ & & & a^{-2}\mathcal{P} \end{pmatrix} \qquad \text{or} \qquad T^\mu_\nu = \begin{pmatrix} -\rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \; . \tag{B.16}$$

For example dust can be described by a perfect fluid with zero pressure, since it is not compressible.

Using our knowledge about the energy-momentum tensor of a perfect fluid, we see that

$$T_{00} = \rho \qquad\qquad T^\mu_\mu = -\rho + 3\mathcal{P} \tag{B.17}$$

and we can rewrite Eqs. (B.10) to obtain the *Friedmann equations*

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{B.18}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3\mathcal{P}\right) \; , \tag{B.19}$$

which are the basic equations of FRW cosmology.

B.3 Continuity Equation

How does the energy-momentum tensor of the perfect fluid evolve with time? In the absence of external forces and gravity, we find that the energy density is constant $\partial\rho/\partial t = 0$ and the Euler equation that the pressure does not depend on the direction $\partial\mathcal{P}/\partial x^i$. In covariant formulation, this amounts to

$$\partial_\mu T^\mu_\nu = 0 \tag{B.20}$$

which some might have seen in the quantum field theory course. The generalisation to general relativity is straightforward by understanding that we have to replace the partial derivative with a covariant derivative to ensure that the continuity equation correctly transforms under a change of coordinates, i.e.

$$0 = \nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha \; . \tag{B.21}$$

For $\nu = 0$ we obtain

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\alpha\mu} T^\alpha_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha = -\frac{\partial\rho}{\partial t} - \Gamma^i_{0i}\rho - \Gamma^i_{0i} T^i_i \tag{B.22}$$

and thus

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot a}{a}\left[\rho + \mathcal{P}\right] = 0 \; . \tag{B.23}$$

B.4 Entropy conservation in thermal equilibrium

For negligible chemical potentials due to number changing processes of photons, e.g. $e^+e^- \to \gamma\gamma, \gamma\gamma\gamma$, we can write the first law of thermodynamics as

$$d(\rho(T)V) = Td(s(T)V) - \mathcal{P}(T)dV \tag{B.24}$$

which allows us to write the entropy density $s(T)$ as

$$s(T) = \frac{\rho(T) + \mathcal{P}(T)}{T} \tag{B.25}$$

by equating the coefficient in front of dV. Similarly it is straightforward to show

$$s(T) = \frac{\partial \mathcal{P}}{\partial T} \tag{B.26}$$

using either one of the Maxwell relations or considering the coefficient in front of the differential VdT in Eq. (B.24). The condition of thermal equilibrium tells us that the entropy in a comoving volume is fixed, i.e.

$$s(T)a^3 = \text{constant} . \tag{B.27}$$

See Dodelson pg. 39/40 for a derivation using the continuity equation.

C The Boltzmann Equation

The Boltzmann equation (for the distribution function) is given by

$$\frac{df}{d\lambda} = C'[f] \tag{C.1}$$

with the distribution function $f = f(\vec{x}, \vec{p}, t)$. The left-hand side gives the change of the distribution function with respect to the affine parameter λ, which we introduced previously and $C'[f]$ is the collision term taking into account any interactions.

We will use the momentum four-vector to define the affine parameter λ as in the section on the geodesic equation. Thus we obtain

$$\frac{df}{dt} = \frac{1}{E}C'[f] = C[f] \tag{C.2}$$

which is exactly Eq. (4.1) in Dodelson[1]. The Boltzmann equations generally connect the different components of the Universe. Electrons and protons are coupled via the Coulomb interaction, photons and electrons[5] via Compton scattering. All particle species are coupled to the metric. See Fig. 12.

C.1 Derivation of the Boltzmann Equation for Number Density

In this section we derive the integrated form of the Boltzmann equation, in particular the Boltzmann equation for the number density. We write the total derivative as the sum of the partial derivatives

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p}\frac{dp}{dt} = \frac{\partial f}{\partial t} - pH\frac{\partial f}{\partial p} \tag{C.3}$$

[5]Compton scattering between protons and photons is suppressed by the larger mass of a proton compared to an electron.

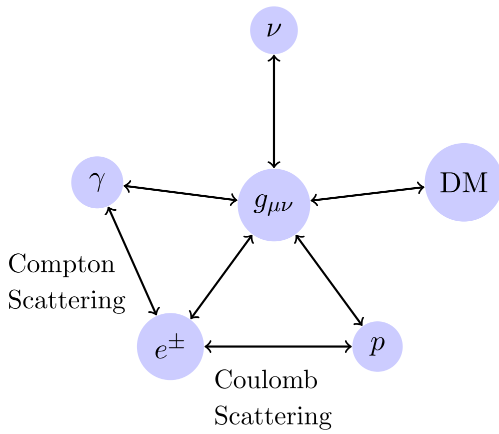


Figure 12: The network of Boltzmann and Einstein equations

using Eq. (A.27). Integrating this equation over the three-momentum we obtain

$$g \int \frac{d^3p}{(2\pi)^3} \frac{df}{dt} = g \int \frac{dp}{(2\pi)^3} 4\pi p^2 \left(\frac{\partial f}{\partial t} - pH \frac{\partial f}{\partial p} \right) \tag{C.4}$$

$$= \frac{d}{dt} g \int \frac{dp^3}{(2\pi)^3} \frac{\partial f}{\partial t} + gH \int \frac{dp}{(2\pi)^3} 4\pi^2 3p^2 f \tag{C.5}$$

$$= \frac{dn}{dt} + 3Hn \tag{C.6}$$

Thus the Boltzmann equation for the number density n_1 of the first particle in case of $2 \to 2$ scatterings, $1 + 2 \leftrightarrow 3 + 4$ is given by

$$a^{-3} \frac{d\left(n_1 a^3\right)}{dt} = \int \prod_{i=1}^{4} d\Pi_i \, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \, |\mathcal{M}|^2$$
$$\times \left\{ f_3 f_4 \left(1 \pm f_1\right) \left(1 \pm f_2\right) - f_1 f_2 \left(1 \pm f_3\right) \left(1 \pm f_4\right) \right\} \tag{C.7}$$

with the phase space integrals

$$\int d\Pi_i = g_i \int \frac{d^3 p_i}{(2\pi)^3 2E_i(p_i)} \tag{C.8}$$

for particle i with g_i internal degrees of freedom.

C.2 Boltzmann equations for linear perturbations

In the following we give a brief outline how to study perturbations with the help of the Boltzmann equation. See Chapter 4 in Dodelson[1] for more detail. We will assume that the perturbations are small and expand all quantities to first order in the small perturbations.

C.2.1 Metric

As we want to study inhomogeneities and anisotropies, we also have to take a perturbation to the metric into account. We will consider perturbations to the flat FRW metric and restrict ourselves to scalar perturbations and do not consider vector or tensor perturbations. The metric in conformal Newtonian gauge is given by

$$ds^2 = -(1 + 2\Psi(\vec{x}, t))dt^2 + a^2(t)(1 + 2\Phi(\vec{x}, t))\delta_{ij}dx^i dx^j \ . \tag{C.9}$$

Ψ is the Newtonian potential and Φ, the perturbation of the spatial curvature. See exercise 3 in chapter 2 of Dodelson[1] to understand better the physical meaning of the two perturbations.

C.2.2 Collisionless Boltzmann Equation for Photons

Not all components of the four-momentum

$$P^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{C.10}$$

are independent. As photons are massless particles, we can use the dispersion relation

$$0 = g_{\mu\nu}P^\mu P^\nu = -(1 + 2\Psi)\left(P^0\right)^2 + p^2 \tag{C.11}$$

with

$$p^2 = g_{ij}P^i P^j = a^2(1 + 2\Phi)\delta_{ij}P^i P^j \tag{C.12}$$

to express the time-component of P^μ in terms of the spatial components

$$P^0 = \frac{p}{\sqrt{1 + 2\Psi}} \doteq p(1 - \Psi) \tag{C.13}$$

to first order in Ψ. Note that an overdense region has $\Psi < 0$ and thus a photon moving out of the potential well will lose energy, i.e. redshift. The independent parameters are thus the position x^i, the momentum p and the direction of the momentum $\hat{p}^i$ satisfying $\hat{p}^i\hat{p}^j\delta_{ij} = 1$. The comoving momentum $P^i = C\hat{p}^i$ are proportional to the momentum direction $\hat{p}^i$. Using Eq. (C.12) we find

$$P^i = p\hat{p}^i \frac{1 - \Phi}{a} \ . \tag{C.14}$$

We write the total derivative as the sum of the partial derivatives

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \ . \tag{C.15}$$

Note that we neglected the partial derivative with respect to $\hat{p}^i$, since it is second order in the small perturbation. Now we have to reexpress the different terms. Using the chain rule we find for

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} \doteq \frac{\hat{p}^i}{a}(1 - \Phi + \Psi) \tag{C.16}$$

to first order. The time derivative of the momentum can be obtained from the zeroth component of the geodesic equation

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \tag{C.17}$$

The left-hand side can be further evaluated to

$$\frac{dP^0}{d\lambda} = p(1 - \Psi)\frac{d}{dt}\left(p(1 - \Psi)\right) = p(1 - \Psi)\left[\frac{dp}{dt}(1 - \Psi) - p\frac{d\Psi}{dt}\right] \tag{C.18}$$

$$= p(1 - \Psi)\left[\frac{dp}{dt}(1 - \Psi) - p\left(\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right)\right] \doteq p(1 - 2\Psi)\left[\frac{dp}{dt} - p\left(\frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right)\right] \ , \tag{C.19}$$

where we used Eq. (C.16). In order to evaluate the right-hand side, we have to evaluate the Christoffel symbol first

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left[\frac{\partial g_{\nu\alpha}}{\partial x^\beta}+\frac{\partial g_{\nu\beta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\frac{P^\alpha P^\beta}{p} = \frac{g^{00}}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta}-\frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p} \tag{C.20}$$

$$= \frac{(-1+2\Psi)}{2}\left\{\frac{\partial g_{00}}{\partial t}\frac{P^0P^0}{p}+\left[2\frac{\partial g_{0i}}{\partial x^j}-\frac{\partial g_{ij}}{\partial t}\right]\frac{P^iP^j}{p}+2\frac{\partial g_{00}}{\partial x^i}\frac{P^0P^i}{p}\right\} \tag{C.21}$$

$$\doteq p\,(1-2\Psi)\left(H+\frac{\partial\Psi}{\partial t}+2\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}+\frac{\partial\Phi}{\partial t}\right) \tag{C.22}$$

Combining everything we obtain

$$\frac{1}{p}\frac{dp}{dt}-\frac{\partial\Psi}{\partial t}-\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = -H-\frac{\partial\Psi}{\partial t}-2\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}-\frac{\partial\Phi}{\partial t} \tag{C.23}$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dt} = -H-\frac{\partial\Phi}{\partial t}-\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\,. \tag{C.24}$$

This equation describes the change in photon momentum. An overdense region has $\Phi > 0$ and $\Psi < 0$. The first term accounts for the redshift due to the expansion, the second term states that a photon loses energy in a deepening potential well and a photon moving into a potential well gains energy.

Finally we have to consider the distribution function. Photons are bosons and thus are described by a Bose-Einstein distribution in equilibrium. We parameterize the perturbation as a change of the variation of the temperature $\Theta(\vec{x},\hat{p},t) = \delta T/T$, i.e.

$$f(\vec{x},p,\hat{p},t) = \left[\exp\left(\frac{p}{T(t)(1+\Theta(\vec{x},\hat{p},t))}\right)-1\right]^{-1}\,. \tag{C.25}$$

Note that we assume that Θ does not depend on the magnitude of the photon momentum. This is a good approximation for small-angle Compton scattering. Thus to leading order we find

$$f = f^{(0)}+\frac{\partial f^{(0)}}{\partial T}T\Theta = f^{(0)}-p\frac{\partial f^{(0)}}{\partial p}\Theta \tag{C.26}$$

using that we can change the differentiation of the exponent from T to p.

After expanding each term in the small perturbations, we can now collect our results in Eqs.(C.15), (C.16), (C.24), and (C.26) and are now able to study the collisionless Boltzmann equation. At zeroth order we obtain

$$0 = \frac{\partial f^{(0)}}{\partial t}-Hp\frac{\partial f^{(0)}}{\partial p} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt}-Hp\frac{\partial f^{(0)}}{\partial p} = -\left(\frac{dT/dt}{T}+\frac{da/dt}{a}\right)p\frac{\partial f^{(0)}}{\partial p} \tag{C.27}$$

and consequently we recover the well-known result $aT = \text{const}$.

At first order we evaluate each of the terms on the right-hand side of Eq. (C.15) separately. The first term evaluates to

$$-p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) = -p\frac{\partial\Theta}{\partial t}\frac{\partial f^{(0)}}{\partial p}-p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p} = -p\frac{\partial\Theta}{\partial t}\frac{\partial f^{(0)}}{\partial p}+\frac{dT/dt}{T}p\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right)\,, \tag{C.28}$$

the second term only consists of one term

$$-p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} \tag{C.29}$$

and the third term

$$H p \Theta \frac{\partial}{\partial p}\left(p \frac{\partial f^{(0)}}{\partial p}\right)-\frac{\partial f^{(0)}}{\partial p} p\left(\frac{\partial \Phi}{\partial t}+\frac{\hat{p}^{i}}{a} \frac{\partial \Psi}{\partial x^{i}}\right) \tag{C.30}$$

partly cancels the first term using the zeroth order equation. Thus we obtain

$$\left.\frac{d f}{d t}\right|_{\text{first order}}=-p \frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t}+\frac{\hat{p}^{i}}{a} \frac{\partial \Theta}{\partial x^{i}}+\frac{\partial \Phi}{\partial t}+\frac{\hat{p}^{i}}{a} \frac{\partial \Psi}{\partial x^{i}}\right] . \tag{C.31}$$

The last two account for the effect of gravity, while the first two are entirely due to the change in the distribution function. It is useful to go to Fourier space for the spatial coordinates and to use conformal time η. Defining

$$\Theta(\vec{x})=\int \frac{d^{3} k}{(2 \pi)^{3}} e^{i \vec{k} \cdot \vec{x}} \tilde{\Theta}(\vec{k}) \tag{C.32}$$

and using a dot to indicate a derivative with respect to conformal time ($\dot{\Theta}=d \Theta / d \eta$) we obtain

$$\left.\frac{d f}{d t}\right|_{\text{first order}}=-\frac{p}{a} \frac{\partial f^{(0)}}{\partial p}\left[\dot{\tilde{\Theta}}+i k \mu \tilde{\Theta}+\dot{\tilde{\Phi}}+i k \mu \tilde{\Psi}\right] \tag{C.33}$$

with $k \mu \equiv \hat{p} \cdot \vec{k}$.

C.2.3 Collision Terms: Compton Scattering

Photons are interacting with electrons via Compton scattering

$$e^{-}(q)+\gamma(p) \leftrightarrow e^{-}\left(q^{\prime}\right)+\gamma\left(p^{\prime}\right) . \tag{C.34}$$

The collision term for Compton scattering is given by

$$C[f(\vec{p})]=\frac{1}{p} \int d \Pi_{q} d \Pi_{q^{\prime}} d \Pi_{p^{\prime}}|\mathcal{M}|^{2}(2 \pi)^{4} \delta^{(4)}\left(p+q-p^{\prime}-q^{\prime}\right)\left(f_{e}(\vec{q}^{\prime}) f(\vec{p}^{\prime})-f_{e}(\vec{q}) f(\vec{p})\right) . \tag{C.35}$$

The integral over $d^{3} q^{\prime}$ is trivially evaluated using the delta function over the three-momenta. We want to evaluate the collision term in the limit of non-relativistic electrons, i.e. $E(\vec{q})=m_{e}+q^{2} / 2 m_{e}$ and consequently also photons with small momenta $|p| \sim T \ll m_{e}$. Thus the energy transfer is

$$E_{e}(\vec{q})-E_{e}(\vec{q}+\vec{p}-\vec{p}^{\prime}) \simeq \frac{(\vec{p}^{\prime}-\vec{p}) \cdot \vec{q}}{m_{e}}, \tag{C.36}$$

where we used that the momentum of the photons is much smaller than the one of the electrons, $|p|,|p^{\prime}| \ll|q|$. As the electrons are non-relativistic, their velocity $q / m_{e} \sim v_{b} \ll 1$ is small and similar to the baryon velocity v_{b}. Thus the energy transfer is of order of $T q / m_{e} \sim T v_{b}$. We can thus expand the energy delta function around zero energy transfer

$$\delta\left(p+\frac{q^{2}}{2 m_{e}}-p^{\prime}-\frac{(\vec{q}+\vec{p}-\vec{p}^{\prime})^{2}}{2 m_{e}}\right) \simeq \delta\left(p-p^{\prime}\right)+\frac{(\vec{p}-\vec{p}^{\prime}) \cdot \vec{q}}{m_{e}} \frac{\partial \delta\left(p-p^{\prime}\right)}{\partial p^{\prime}} . \tag{C.37}$$

Similarly we can make the approximation that the momentum distribution of the electrons does not change, i.e. $f_{e}(\vec{q}+\vec{p}-\vec{p}^{\prime}) \simeq f_{e}(\vec{q})$. Thus the collision term reads

$$C[f(\vec{p})]=\frac{\pi}{4 m_{e}^{2} p} \int \frac{d^{3} q}{(2 \pi)^{3}} f_{e}(\vec{q}) \int \frac{d^{3} p^{\prime}}{(2 \pi)^{3} p^{\prime}}|\mathcal{M}|^{2}\left(\delta\left(p-p^{\prime}\right)+\frac{(\vec{p}-\vec{p}^{\prime}) \cdot \vec{q}}{m_{e}} \frac{\partial \delta\left(p-p^{\prime}\right)}{\partial p^{\prime}}\right)\left(f(\vec{p}^{\prime})-f(\vec{p})\right) \tag{C.38}$$

Following Dodelson, we will neglect the polarisation of the photons and average over it for simplicity of the discussion. A proper treatment should include the polarisation. The matrix element for Compton scattering after summing over the polarisation of the incoming and outgoing photons is given by

$$|\mathcal{M}|^2 = 4\pi\sigma_T m_e^2 \left(2 + P_2(\hat{p}\cdot\hat{p}')\right) = 4\pi\sigma_T m_e^2 \left(2 + \frac{4\pi}{5}\sum_{m=-2}^{2} Y_{2m}(\hat{p})Y_{2m}^*(\hat{p}')\right) \tag{C.39}$$

with the second Legendre polynomial $2P_2(x) = 3x^2 - 1$ and the corresponding spherical harmonics Y_{lm}. Note that the integral over the azimuthal angle φ vanishes for $Y_{lm}(\hat{p}') \propto e^{im\varphi}$ for $m \neq 0$. Thus we find using $Y_{20}(\hat{p}) = \sqrt{5/4\pi}P_2(\hat{p}\cdot\hat{k})$ for some fixed $\hat{k}$

$$|\mathcal{M}|^2 = 4\pi\sigma_T m_e^2 \left(2 + P_2(\mu)P_2(\mu')\right) \tag{C.40}$$

with $\mu' = \hat{p}'\cdot\hat{k}$ The q-integration yields the number density n_e and the velocity $n_e\vec{v}_b$ in case of the factor $\vec{q}$. We expand the distribution functions (C.26)

$$f(p') - f(p) = f^{(0)}(p') - f^{(0)}(p) - p'\frac{\partial f^{(0)}(p')}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}(p)}{\partial p}\Theta(\hat{p}) \ . \tag{C.41}$$

It is easy to see that the zeroth order term vanishes, because the product of the delta-function with the difference of the equilibrium distributions vanishes, when integrated over p'. The leading non-vanishing term is given by the sum of the terms proportional to v_b and Θ, which we evaluate separately. Using spherical coordinates for the p' integral we obtain

$$C_\Theta[f(\vec{p})] = \frac{\sigma_T n_e}{8\pi p}\int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right)\int p'dp'\delta(p-p')\left(-p'\frac{\partial f^{(0)}(p')}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}(p)}{\partial p}\Theta(\hat{p})\right) \tag{C.42}$$

$$. \tag{C.43}$$

The momentum integral of C_Θ can be directly evaluated with the delta function and we obtain

$$C_\Theta[f(\vec{p})] = \sigma_T n_e p\frac{\partial f^{(0)}(p)}{\partial p}\left(\Theta(\hat{p}) - \Theta_0 + \frac{1}{2}P_2(\mu)\Theta_2\right) \ , \tag{C.44}$$

where we defined the moments

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu'}{2}P_l(\mu')\Theta(\mu') \ . \tag{C.45}$$

The second term C_{v_b}

$$C_{v_b}[f(\vec{p})] = \frac{\sigma_T n_e}{8\pi p}\int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right)\int p'dp'(\vec{p}-\vec{p}')\cdot\vec{v}_b\frac{\partial\delta(p-p')}{\partial p'}\left(f^{(0)}(p') - f^{(0)}(p)\right) \tag{C.46}$$

can be similarly evaluated by first noticing that the angular integral over $P_2(\mu')\vec{p}'\cdot\vec{v}_b$ is odd and vanishes and by using partial integration to remove the derivative from the delta function

$$\begin{aligned} C_{v_b}[f(\vec{p})] &= -\frac{\sigma_T n_e}{8\pi p}\int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right)\int dp'\delta(p-p')\frac{\partial}{\partial p'}\left[p'\vec{p}\cdot\vec{v}_b\left(f^{(0)}(p') - f^{(0)}(p)\right)\right] && \text{(C.47)} \\ &= -\frac{\sigma_T n_e}{8\pi p}\int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right)\int dp'\delta(p-p')\left[p'\vec{p}\cdot\vec{v}_b\frac{\partial f^{(0)}(p')}{\partial p'}\right] && \text{(C.48)} \\ &= -\sigma_T n_e p\frac{\partial f^{(0)}(p)}{\partial p}\hat{p}\cdot\vec{v}_b \ . && \text{(C.49)} \end{aligned}$$

Collecting both term, we obtain for the collision term

$$C[f(\vec{p})] = \sigma_T n_e p \frac{\partial f^{(0)}(p)}{\partial p} \left[\Theta(\hat{p}) - \Theta_0 + \frac{1}{2} P_2(\mu)\Theta_2 - \hat{p}\cdot\vec{v}_b \right] . \tag{C.50}$$

Finally we take the Fourier transform of the collision term. Typically we will assume that the velocity points in the same direction as $\vec{k}$, i.e. so the Fourier transform of $\hat{p}\cdot\vec{v}_b$ becomes $\tilde{v}_b\mu$. Hence the Fourier transform (of the spatial directions) is given by

$$C[f(\vec{p})] = \sigma_T n_e p \frac{\partial f^{(0)}(p)}{\partial p} \left[\tilde{\Theta}(\hat{p}) - \tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 - \tilde{v}_b\mu \right] \tag{C.51}$$

C.2.4 Boltzmann Equation for Photons

Combining the result for the collisionless Boltzmann equation (C.33) with the collision term (C.51), we obtain the Boltzmann equation for photons coupled to non-relativistic electrons

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = \sigma_T n_e a \left[\tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 + \tilde{v}_b\mu - \tilde{\Theta} \right] . \tag{C.52}$$

Note that the different Fourier modes do not mix and thus evolve independently. This only holds in the linear regime. If the perturbations can be large, as it is the case for matter, the linear approximation breaks down and different Fourier modes will couple. Finally we use the optical depth

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a \ , \tag{C.53}$$

which characterises how much light is absorbed by the electrons, i.e. the intensity $I(\eta_0)$ today compared to the intensity at η ,

$$I(\eta_0) = I(\eta) e^{-\tau} \tag{C.54}$$

to write the Boltzmann equation describing photons

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau} \left[\tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 + \tilde{v}_b\mu - \tilde{\Theta} \right] . \tag{C.55}$$

References

[1] S. Dodelson, *Modern Cosmology* (2003).

[2] D. Baumann, *Cosmology*, `http://www.damtp.cam.ac.uk/user/db275/Cosmology.pdf`, accessed: 2016-08-02.

[3] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, 1990), frontiers in Physics, 69.

[4] E. Hubble, Proceedings of the National Academy of Science **15**, 168 (1929).

[5] P. A. R. Ade et al. (Planck) (2015), arXiv: `1502.01589`.

[6] P. Cushman et al., in *Community Summer Study 2013: Snowmass on the Mississippi (CSS2013) Minneapolis, MN, USA, July 29-August 6, 2013* (2013), arXiv: `1310.8327`.