# PHYS3115 - Particle Physics and the Early Universe –particle physics–

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Particle physics is about the description of the fundamental building blocks of nature: its elementary particles and forces. It has close connections with the evolution of the universe at the earliest times, because the universe used to be much hotter and thus particles were more energetic and have been copiously produced. This is one of the reasons for a combined course on particle physics and the early universe.

The underlying theoretical framework of particle physics is quantum field theory. This course is not a course on quantum field theory which is the subject of the Honours level quantum field theory course, but I will touch on some aspects of quantum field theory. This course serves as an introduction to the bigger picture of particle physics. It will mostly focus on the theoretical side, but also contain some aspects of the experimental success of the Standard Model.

A good book to read while following this course is Mark Thompson, Modern Particle Physics which is based on a Part III (year 4) course taught by the author at Cambridge.

These notes are partly based on the A/Prof Yvonne Wong's lecture notes, who taught the course in previous years.

#### Conventions

Throughout the course we will use natural units

$$\hbar = c = k_B = 1$$

and the signature (+ - - -) for the metric. Thus length scales and time are measured with the same units

$$[\ell] = [t] = \frac{1}{[m]} = \frac{1}{[E]} = \frac{1}{[T]} = eV^{-1}$$

which is the inverse of energy. Temperature is measured in the same units as energy. The relativistic energy-momentum relation reads

$$p^2 = p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2 \tag{0.1}$$

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## 1 Introduction and preliminaries

Particle physics addresses the questions

- What are the elementary building blocks of nature?
- What is matter made of at the most fundamental level?
- How do the building blocks interact with each other?

In order to probe physics at the shortest distances, we have to use the highest energies. This concept is summarised in the de Broglie wavelength of a particle in quantum mechanics

$$\lambda = \frac{h}{|\mathbf{p}|} \,. \tag{1.1}$$

Fig. 1 illustrates the length scales which we are dealing with in particle physics. The energies are measured in eV, where 1eV equals  $1.6 \times 10^{-19} J$ , the energy gained/lost by an electron across a potential difference of 1V. Molecules and solids are built up by atoms, which themselves have a nucleus and electrons. The nucleus is made of protons and neutrons, so-called nucleons, which are held together by the exchange of mesons. The nucleons belong to the larger family of baryons, bound states of three quarks, while mesons are bound states of a quark and an anti-quarks. Quarks interact via the strong force and electrons are leptons, which do not interact via the strong force.

According to today's knowledge up to energies of  $\text{TeV} = 10^{12} \text{ eV}$  or length scales of  $10^{-18}$  cm, quarks and leptons do not possess any substructure and are the elementary building blocks of nature.



Figure 1: Comparison of different length scales. Taken from CERN

Apart from the elementary particles which constitute all ordinary matter (protons, neutrons, electrons), there are further heavier particles, which are generally unstable. They have been observed in cosmic rays, e.g. the muon, a heavy version of the electron, discovered in 1936 by Anderson and Neddermeyer while studying cosmic radiation (Nobel prize 1936) or the  $\tau$  lepton, which has been predicted by Yung-su Tsai in 1971 and detected in experiments between 1974 and 1977 by Martin L. Perl (Nobel prize, Martin L. Perl 1995) and colleagues at the Stanford Linear Accelerator Center (SLAC) and the Lawrence Berkeley National Laboratory (LBL).

Experimental research in this field is often split in three different categories: the energy frontier, precision or intensity frontier and the cosmological frontier.

- 1. Typical experiments at the energy frontier are colliders, like the Large Hadron Collider (LHC), which accelerate particles to relativistic energies and then collide them to study physics at the highest energies. Typical analyses look at scattering of particles (deflection angle, when two particles collider) and decays of particles produced in collisions (debris left behind when a particle disintegrates). Cosmic radiation also probes the highest energies, even larger than the ones accessible at the LHC, but with much smaller fluxes, interaction rates.
- 2. At the precision/intensity frontier, experiments such as flavour physics experiments like Belle 2 in Japan, neutrino oscillation experiments, dark matter direction detection experiments, and others make high precision measurements at much lower energies. This allows to probe very weakly interacting physics like dark matter and neutrinos, probe for very rare processes such as b → sγ. Through their high precision they are sensitive to small (quantum) corrections due to new physics.
- 3. Finally, the cosmology frontier uses our whole Universe to test the Standard Model (SM) and search for new physics using observations of big bang nucleosynthesis, the cosmic microwave background and other observables.

When combining quantum mechanics and special relativity, one naturally arrives at quantum field theory (QFT), which is the underlying framework of particle physics. Quantum field theory does not specify the particles and interactions itself. It only ensures that the introduced particles and interactions are consistent with special relativity and quantum physics. This is similar to Newton's second law in mechanics which does not specify the force. Particles are described by (quantum) excitations of a field, e.g. the photon is an excitation of the electromagnetic field. Our current understanding of particle physics is summarised in the SM of particle physics which describes nature (apart from a few deviations which we will comment on later). The Lagrangian which describes all interactions in the SM fits on a mug, which is shown on the title page. Before discussing the SM, we introduce natural units.

### 1.1 Natural units

In particle physics it is common to use natural units instead of SI units. This amounts to replacing basic SI units [kg,m,s,K] for mass, length, time and temperature with  $[\hbar, c, \text{GeV}, k_B]$ . Thus we find

	$\mathbf{SI}$	$[\hbar,c,{\rm GeV}]$	$\hbar = c = k_B = 1$
energy	${\rm kg}m^2s^{-2}$	$\mathrm{GeV}$	$\mathrm{GeV}$
momentum	${ m kg}ms^{-1}$	${\rm GeV/c}$	$\mathrm{GeV}$
mass	kg	${ m GeV}/c^2$	$\mathrm{GeV}$
time	S	$({\rm GeV}/\hbar)^{-1}$	${\rm GeV}^{-1}$
length	m	$(\text{GeV}/\hbar c)^{-1}$	${\rm GeV}^{-1}$
temperature	Κ	$\mathrm{GeV}/k_B$	$\mathrm{GeV}$

By setting  $\hbar = c = k_B = 1$  we can express the units of all quantities in powers of GeV which is extremely convenient <sup>1</sup>. It is straightforward to convert between SI units and natural units using the expressions for  $\hbar$ , c, and  $k_B$ 

$$1 \,\text{GeV} = 1.602 \times 10^{-10} \, J = 1.602 \times 10^{-10} \,\text{kg} \, m^2 \, s^{-2} \tag{1.2}$$

$$\hbar = 6.582 \times 10^{-25} \,\text{GeV}\,s \tag{1.3}$$

$$c = 2.998 \times 10^8 \, m \, s^{-1} \tag{1.4}$$

$$\hbar c = 0.197 \times 10^{-15} \,\text{GeV}\,m \tag{1.5}$$

$$k_B = 8.617 \times 10^{-5} \,\mathrm{eV} \, K^{-1} \,. \tag{1.6}$$

For example for the root-mean-square charge radius of the proton

$$\langle r^2 \rangle^{1/2} = 4.1 \text{GeV}^{-1}$$
 (1.7)

can be converted to m by multiplying with the correct number of factors of  $\hbar$  and c to obtain m

$$\langle r^2 \rangle^{1/2} = 4.1 \text{GeV}^{-1} \hbar c = 0.81 \times 10^{-15} \, m \,.$$
 (1.8)

Another common choice in cosmology is to introduce the reduced Planck mass

$$m_{\rm Planck} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.435 \times 10^{18} \,{\rm GeV}$$
 (1.9)

which will be discussed more in the second half of the course.

#### **1.2** Standard Model of particle physics

The Standard Model (SM) of particle physics has been first formulated in the 1960's and finalised in its current form in the mid 1970's. It passed essentially all experimental tests and makes predictions with

<sup>&</sup>lt;sup>1</sup>The conversion from Heavyside-Lorentz units to natural units requires to set  $\epsilon_0 = \mu_0 = 1$ , where  $\epsilon_0$  and  $\mu_0$  are the permittivity and permeability in vacuum.



Figure 2: The Standard Model of particle physics.

high accuracy. The anomalous magnetic moment of the electron agrees with the SM prediction to more than 12 significant figures precision and is one of the most precisely known quantities. Fig. 2 illustrates the particle content of the SM of particle physics. A good source for details about elementary particles is the website of the Particle Data Group http://pdg.lbl.gov.

The particles can be separated into spin  $\frac{1}{2}$  fermions, the forces which are mediated by the spin 1 gauge bosons and the spin 0 Higgs boson. Gravity is mediated by a spin 2 graviton. It is not part of the SM of particle physics and one of the open questions in theoretical particle physics.

The SM describes three different forces:

- the strong force, which is mediated by gluons and responsible for forming protons, neutrons, and more generally hadrons, composite states of multiple quarks and anti-quarks. We distinguish baryons which are composite states of three quarks and mesons which are composite states of a quark anti-quark pair;
- 2. the electromagnetic force, which is mediated by the photon;
- 3. and the weak nuclear force, which is responsible for beta decay

While in a classical theory (e.g. electromagnetism), a source charge will bend the trajectory of a test charge via the Coulomb interaction through an action at a distance (see Fig. 3a), in a quantum theory, the different particles interact by exchanging a photon (the force carrier of the electromagnetic interaction), see Fig. 3b. Either the electron or the positron in the shown figure emit a *virtual* 



Figure 3: Scattering

photon with some momentum  $\Delta \mathbf{p}$ . This photon is absorbed by the other particle at a distance  $\Delta \mathbf{x}$ and thus changes its momentum. The virtual photon mediates the interaction between the electron and positron and is the force carrier of electromagnetism. It is a virtual particle, because it does not respect Einstein's relation  $E^2 = \mathbf{p}^2 + m^2$ . As a photon is simply a quantized electromagnetic wave, and electromagnetic waves are oscillating  $\mathbf{E}$  and  $\mathbf{B}$  field solutions of Maxwell's equations. Electromagnetic fields are vector fields which corresponds to spin 1 in the quantum view of the photon. A similar discussion applies for the other force carriers (gluons and electroweak force).

The spin  $\frac{1}{2}$  fermions constitute all matter and can be classified according to how they interact via the different forces: The quarks, the top-half in the figure, couple to gluons and interact via the strong interaction, while leptons do not couple to gluons. Each particle is accompanied by its anti-particle, which has the same mass, but opposite charges. See the discussion of the Dirac equation.

The quarks can be further separated into up-type quarks with electric charge 2/3e (first row) and down-type quarks with electric charge -1/3e (2 second row). In nature we do not see elementary light quarks [u, d, s, c, b], because they are always confined in composite states, the so-called hadrons. We distinguish between mesons, composite states of one quark and one anti-quark and baryons, composite states of three quarks. There are two different types of leptons: Charged leptons with electric charge -1e (fourth row) and neutrinos which do not carry any electric charge (third row).

For each type of fermion, there are three copies, which are distinguished by their mass. These copies are commonly called generations, families or flavours. The first generation is made up of the lightest particles, the up (u) and down (d) quarks as well as the electron neutrino  $(\nu_e)$  and the electron  $(e^-)$ . The up and down quark form the building block for protons p = (uud) and neutrons n = (udd). Protons and neutrons form nuclei, which form atoms together with electrons. The fermion masses increase when going to the second and third generation.

The SM predicts neutrinos to be massless. However in 1998 Super-Kamiokande measured atmospheric neutrino oscillations and showed that neutrinos have a tiny mass. The absolute mass scale is not known, but restricted to be smaller than about 0.1 - 1 eV. Neutrino masses are one evidence of physics beyond the SM.

Finally, the Higgs boson is a spin 0 particle, i.e. it remains the same under Lorentz transformations. It is required in the SM to give mass to all fermions  $^2$  and gauge bosons. At high temperatures and/or

<sup>&</sup>lt;sup>2</sup>The mass mechanism for neutrinos is still unknown.



Figure 4: Electroweak phase transition and the Higgs potential  $V_T(\phi_c)$  at finite temperature as a function of the Higgs field  $|\phi_c|$ .

high energies <sup>3</sup> the symmetry of the SM is enhanced and the electromagnetic force and the weak nuclear force are unified in the electroweak force, i.e. they are effectively the same force. Below the critical temperature, there is a phase transition (the electroweak phase transition), this symmetry is broken and the W, Z boson and all the fermions obtain masses. See Fig. 4 for an illustration of the temperature dependence of the Higgs potential. The Higgs boson has postulated in 1964 and discovered in 2012 at the Large Hadron Collider (LHC).

#### 1.3 Unsolved questions

There are several open questions in particle physics and in the intersection with cosmology<sup>4</sup>

- 1. What is the mass mechanism for neutrinos?
- 2. Why are there three generations?
- 3. Is there any explanation for the large hierarchy among the fermion masses?
- 4. What is the particle physics description of dark matter?

<sup>&</sup>lt;sup>3</sup>At high energies  $E \gg v = (\sqrt{2}G_F)^{-1/2} \simeq 246$  GeV ( $G_F$  is the Fermi constant), the effects of the symmetry breaking which are quantified by the order parameter v are small. In other words, the symmetry is enlarged in the limit  $v \to 0$ .

<sup>&</sup>lt;sup>4</sup>Cosmology studies the evolution of the universe from very early times to today. According to the big bang model, as the universe was very hot and then slowly cooled down, the reactions in the early universe are described by particle physics. Thus particle physics gives insight into cosmology and particle physics is probed by cosmological measurements of early times. This leads to a close connection between the physics at the shortest distances with physics at the longest distances.

- 5. Why is there more matter than anti-matter? What is the mechanism of baryogenesis?
- 6. What is correct particle physics description of dark energy? How does it relate to the vacuum energy predicted by quantum field theory?
- 7. Which scalar field drives inflation?

#### 1.4 Special relativity

The section provides a brief recap of special relativity. Special relativity is based on two postulates

- 1. The laws of physics are the same in all inertial reference frames.
- 2. The speed of light (in vacuum) is the same in all inertial reference frames.

Consider the emission and absorption of a light pulse in two different inertial reference frames. In the first inertial reference frame I, the light is emitted from a source S at position  $\mathbf{x}_S$  time  $t_S$  and absorbed by a detector D at position  $\mathbf{x}_D$  and time  $t_D$ . Similarly, in the second inertial reference frame the light is emitted at  $(\mathbf{x}'_S, t'_S)$  and absorbed at  $(\mathbf{x}'_D, t'_D)$ . As the laws of physics are the same in both reference frames, the light is propagating along straight lines from the source to the detector in both frames and also the emission and absorption work in the same way. Moreover, light is propagating at the speed of light c and thus we find<sup>5</sup>

$$|\mathbf{x}_D - \mathbf{x}_S| = c(t_D - t_S) \qquad |\mathbf{x}'_D - \mathbf{x}'_S| = c(t'_D - t'_S).$$
(1.10)

This motivates the definition of a distance between two points in spacetime  $(t, \mathbf{x})$ 

$$\Delta s^2 \equiv c^2 (t_D - t_S)^2 - (\mathbf{x}_D - \mathbf{x}_S)^2 .$$
 (1.11)

which is the same in all inertial reference frames  $\Delta s^2 = \Delta s'^2$ . This condition can be expressed very efficiently using 4-vector notation  $x^{\mu} = (t, \mathbf{x})$ 

$$\Delta s^{2} = (x_{D} - x_{S})^{\mu} (x_{D} - x_{S})_{\mu} = \eta_{\mu\nu} (x_{D} - x_{S})^{\mu} (x_{D} - x_{S})^{\nu} \qquad (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
(1.12)

where we implicitly defined the 4-vector  $x_{\mu} \equiv \eta_{\mu\nu} x^{\nu} = (t, -\mathbf{x})$  with a lower index and used the Einstein sum convention that repeated upper and lower indices are summed over. The quantity  $\eta_{\mu\nu}$  is the *Minkowski metric* which can be used to lower indices. The inverse of the Minkowski metric  $\eta^{\mu\nu}$  can be used to raise indices. Lorentz transformations from a frame I to a frame I'

$$x^{\prime\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \tag{1.13}$$

<sup>&</sup>lt;sup>5</sup>Causality ensures that the time difference is positive in both cases.

leave  $\Delta s^2$  invariant. For example a Lorentz boost along the x-direction is given by

$$(\Lambda^{\mu}_{\nu}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$
(1.14)

with  $\beta = v/c$  and  $\gamma = 1/\sqrt{1-\beta^2}$ . The upper index  $\mu$  labels the rows and the lower index  $\nu$  the columns of the matrix. The inverse Lorentz transformation is obtained by replacing  $\beta \to -\beta$ .

Any combination of 4-vectors  $x^{\mu}$  and  $y^{\mu}$ 

$$\eta_{\mu\nu}x^{\mu}y^{\nu} \tag{1.15}$$

is invariant under Lorentz transformations and thus the same in all inertial reference frames, i.e. the metric  $\eta_{\mu\nu}$  defines a scalar product. The set (more precisely vector space) of 4-vectors together with the Minkowski metric defines Minkowski space. 4-vectors with upper indices are called *contravariant* and 4-vectors with lower indices *covariant*.

4-momentum: Apart from 4-vectors in coordinate space, we can define other 4-vectors. Energy and 3-momentum can be combined in a 4-vector in special relativity

$$p^{\mu} = (E, \mathbf{p}) . \tag{1.16}$$

The scalar product  $p_{\mu}p^{\mu} = E^2 - \mathbf{p}^2$  is invariant under Lorentz transformations and for a single particle it is simply given by in terms of the (rest) mass m of the particle,  $p_{\mu}p^{\mu} = m^2$ .

4-derivative: We can define the four-derivative

$$\frac{\partial}{\partial x^{\mu}} \equiv \left(\frac{\partial}{\partial t}, \nabla\right) \tag{1.17}$$

How does it transform under Lorentz transformations? Using the explicit definition of the 4-derivative we obtain

$$\sum_{\mu} \frac{\partial}{\partial x^{\mu}} x^{\mu} = 4 , \qquad (1.18)$$

which is the same constant independent of the inertial reference frame and thus a Lorentz scalar. As  $x^{\mu}$  is a contravariant 4-vector,  $\partial/\partial x^{\mu}$  has to be a covariant vector with a lower index. We thus define

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \,. \tag{1.19}$$

Using the inverse metric, we can define the contravariant derivative

$$\partial^{\mu} = \eta^{\mu\nu} \partial_{\nu} = \left(\frac{\partial}{\partial t}, -\nabla\right) \,. \tag{1.20}$$



Figure 5: Normalization volume

## 2 Cross sections

In order to calculate the cross section, we have to calculate the probability for a transition from state  $|i\rangle$  to state  $|f\rangle$ 

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} .$$
(2.1)

For definiteness we will consider the scattering of 2 particles to an arbitrary final state with n particles. In the infinite past and future (in the absence of long-range interactions) an n-particle state can be written as the product of n 1-particle states, e.g. the two-particle state  $|p_1p_2\rangle = |p_1\rangle |p_2\rangle$ . In order to evaluate the probability, we require the normalization of a 1-particle state. In particle physics, we usually work with momentum eigenstates, which are described by plane waves. The quantization boundary conditions will only allow discrete set of wave vectors. Now, if we consider the wave function of a particle in a box of volume V and length L as shown in Fig. 5, a Lorentz boost in x direction with  $\beta$  leads to a Lorentz contraction by  $\gamma = 1/\sqrt{1-\beta^2}$ .

In order to have a Lorentz-invariant normalization, the normalization of a one-particle state in QFT is

$$\langle k|k\rangle = 2k^0 V , \qquad (2.2)$$

where  $k^0$  denotes the energy and V the volume, in which we quantize the particle. It is proportional to the energy  $k^0$  in order to be Lorentz invariant. It is proportional to volume, because the states are plane waves with definite momentum. As the integral over the square of a plane wave scales with the integration volume V, the scalar product  $\langle k|k\rangle$  is proportional to V.

As the two-particle state in the infinite past and future can be effectively described by the product of two one-particle states, we obtain

$$\langle i|i\rangle = 4E_1 E_2 V^2 \qquad \langle f|f\rangle = \prod_i (2k_i^0 V) , \qquad (2.3)$$

where  $E_1$  and  $E_2$  are the energies of the incoming particles and  $k_i^0$  the energies of the outgoing particles.

Turning to the S-matrix element  $\langle f|i\rangle$ , which describes the scattering of a state  $|i\rangle$  which is defined at  $t \to -\infty$  to a state  $|f\rangle$  at time  $t \to \infty$ . The S matrix is usually split into the forward scattering part with  $|f\rangle = |i\rangle$  and the transfer matrix element  $i\mathcal{T}_{fi} = \langle f|i\rangle$  with  $|f\rangle \neq |i\rangle$ . We are only interested in non-trivial scattering which is described by the transfer matrix element  $\mathcal{T}_{fi}$ . If the theory is translation invariant both in space and time, 4-momentum will be conserved and thus we can write the transfer matrix element as  $i\mathcal{T}_{fi} = (2\pi)^4 \delta^{(4)} (k_{in}^{\mu} - k_{out}^{\mu}) i\mathcal{M}$ , where we factor out a delta function which ensures 4-momentum conservation and  $\mathcal{M}$  is the Lorentz-invariant matrix element. Thus for the squared transition amplitude with  $|f\rangle \neq |i\rangle$  we obtain

$$|\langle f|i\rangle|^{2} = |(2\pi)^{4}\delta^{(4)}(k_{in}^{\mu} - k_{out}^{\mu})|^{2}|\mathcal{M}|^{2} = (2\pi)^{4}\delta^{(4)}(k_{in}^{\mu} - k_{out}^{\mu})VT|\mathcal{M}|^{2}, \qquad (2.4)$$

where we used  $(2\pi)^4 \delta^{(4)}(0) = \int d^4x e^{i0} = VT$ . Finally we have to sum over all possible final states. In the box with length L the 3-momenta are quantized  $\mathbf{k_i} = \frac{2\pi}{L} \mathbf{n_i}$  and thus summing over the different modes corresponds to an integration

$$\sum_{n_i} \to \frac{L^3}{(2\pi)^3} \int d^3 k_i .$$
 (2.5)

The volume  $V = L^3$  cancels against the volume factor from the normalization. Thus the transition rate is

$$\dot{P} = \frac{(2\pi)^4 \delta^{(4)}(k_{in} - k_{out})}{4E_1 E_2 V} \int \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3 2k_i^0} |\mathcal{M}|^2 \,.$$
(2.6)

Let us denote the incoming momenta by  $p_1$  and  $p_2$  and the outgoing momenta by  $k_i$ . The differential cross section can be obtained by dividing by the incoming particle flux. In the rest frame of the second particle, it is simply given by the velocity of the first particle per volume. In the centre-of-mass frame (where the 3-momenta of the incoming particles add to zero), it is the relative velocity per volume  $\sigma = \dot{P}V/v_{rel}$  where the relative velocity can be expressed by

$$v_{rel} = |\mathbf{v_1} - \mathbf{v_2}| = \left|\frac{\mathbf{p_1}}{E_1} - \frac{\mathbf{p_2}}{E_2}\right| = \frac{|\mathbf{p_1}|}{E_1 E_2} (E_1 + E_2) = \frac{|\mathbf{p_1}|}{E_1 E_2} \sqrt{s}$$
(2.7)

In the last equation we introduced the so-called Mandelstam variable  $s = (p_1 + p_2)^2$ , which is Lorentzinvariant by construction. The differential cross section can also be written in terms of the flux factor  $F = E_1 E_2 v_{rel}$ . In order to simplify the expression, note that in the centre-of-mass frame of a 2-particle system the 3-momenta add to zero  $\mathbf{p_1} + \mathbf{p_2} = 0$  and we can express the energy and momentum of the particles in terms of its masses and the Mandelstam variable  $s = (p_1^{\mu} + p_2^{\mu})^2 = (E_1 + E_2)^2$ , where  $p_i$ are the 4-momenta. We can rewrite the Mandelstam variable s as follows

$$s = (p_1^{\mu} + p_2^{\mu})^2 = m_1^2 + m_2^2 + 2p_1^{\mu}p_{2\mu}$$
(2.8)

$$= m_1^2 + m_2^2 + 2E_1E_2 - 2\mathbf{p_1} \cdot \mathbf{p_2}$$
(2.9)

$$= m_1^2 + m_2^2 + 2E_1E_2 + 2|\mathbf{p}_1|^2 \tag{2.10}$$

$$= m_1^2 + m_2^2 + 2E_1E_2 + 2E_1^2 - 2m_1^2$$
(2.11)

$$= m_2^2 - m_1^2 + 2E_1(E_2 + E_1) = m_2^2 - m_1^2 + 2E_1\sqrt{s}$$
(2.12)

and thus find for the energy of one particle in the centre-of-mass frame

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} . \tag{2.13}$$

Using the expression for  $E_1$  we find for the 3-momentum  $|\mathbf{p_1}|$ 

$$|\mathbf{p_1}| = \sqrt{E_1^2 - m_1^2} = \frac{1}{2\sqrt{s}}\sqrt{(s + m_1^2 - m_2^2)^2 - 4sm_1^2}$$
(2.14)

$$=\frac{1}{2\sqrt{s}}\sqrt{s^2+m_1^4+m_2^4-2sm_1^2-2sm_2^2-2m_1^2m_2^2}$$
(2.15)

$$=\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}} \tag{2.16}$$

which can be expressed in terms of the Källén function  $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ . Thus we obtain for the flux factor

$$F = E_1 E_2 v_{rel} = \sqrt{s} |\mathbf{p_1}| = \frac{1}{2} \lambda^{1/2} (s, m_1^2, m_2^2)$$
(2.17)

and can express the differential cross section as

$$4Fd\sigma = |\mathcal{M}|^2 d\text{LIPS}_n(p_1 + p_2) , \qquad (2.18)$$

where the Lorentz-invariant n-body phase space is defined by

$$d\text{LIPS}_n(k) \equiv (2\pi)^4 \delta^{(4)} \left( k - \sum_{i=1}^n k_i \right) \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3 2k_i^0} \,. \tag{2.19}$$

The 2-body phase space is particularly simple to evaluate in the centre-of-mass frame

$$\int d\text{LIPS}_2(k) = \int (2\pi)^4 \delta^{(4)}(k - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \frac{d^3 k_2}{(2\pi)^3 2k_2^0}$$
(2.20)

$$= \int (2\pi)\delta(\sqrt{s} - k_1^0(|\mathbf{k_1}|) - k_2^0(|\mathbf{k_1}|)) \frac{d^3k_1}{(2\pi)^3 2k_1^0 2k_2^0}$$
(2.21)

$$= \int \delta(\sqrt{s} - k_1^0 - k_2^0(k_1^0)) \frac{|\mathbf{k}_1| dk_1^0 d\Omega}{16\pi^2 k_2^0(k_1^0)}$$
(2.22)

$$= \int \frac{d\Omega}{16\pi^2} \frac{|\mathbf{k}_1|}{\sqrt{s}} \tag{2.23}$$

$$=\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{32\pi^2 s}\int d\Omega$$
(2.24)

$$=\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{16\pi s}\int_{-1}^1 d\cos\theta$$
(2.25)

with the Källén function  $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ . In order to evaluate the energy integral over  $k_1^0$ , we used

$$\delta(f(x)) = \sum_{\text{zeros}x'} \frac{\delta(x-x')}{|f'(x')|} .$$
(2.26)

For the second-last line we used that  $|\mathbf{k_1}|$  can be expressed in terms of the Källén function. In the last line we used that the squared matrix element for a 2-body final state generally does not depend on the azimuthal angle  $\phi$  in the centre-of-mass frame. It is convenient to express the result in terms of Lorentz-invariant quantities. Scattering of 2 particles into 2 particles can be described in terms of 3 Lorentz-invariant quantities, the so-called *Mandelstam variables* 

$$s = (p_1 + p_2)^2$$
  $t = (p_1 - k_3)^2$   $u = (k_1 - k_4)^2$ . (2.27)

The Mandelstam variables satisfy  $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$ , where  $m_i$  are the masses of the initial and final state particles.

In case that there are n identical particles in the final state, we have to divide by the symmetry factor  $S_F = n!$  in the cross section in order to avoid double-counting the different final state configurations. Thus the total cross section is given by

$$\sigma = \frac{1}{S_F} \int d\sigma \;. \tag{2.28}$$

Hence the  $2 \rightarrow 2$  scattering cross section  $1 + 2 \rightarrow 3 + 4$  in the centre of mass frame is given by

$$\sigma = \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{64\pi \ sF} \frac{1}{S_F} \int_{-1}^1 |\mathcal{M}|^2 d\cos\theta \ . \tag{2.29}$$

The cross section for example can be used to determine the number of particles produced at a collider: For example the cross section to produce a Z boson for  $\sqrt{s} = m_Z = 91$  GeV is given by  $\sigma = 4 \times 10^4 pb$ . The Large Electron Positron (LEP) collider at CERN had an instantaneous luminosity of  $\mathcal{L} = 10^{31} \text{cm}^{-2} s^{-1}$ . One year has roughly  $3 \times 10^7 s$ . Thus LEP produced about

$$\sigma \mathcal{L}t = 4 \times 10^4 pb \times 10^{31} \text{cm}^{-2} s^{-1} \times 3 \times 10^7 s \approx 10^7$$
(2.30)

Z bosons per year, where we used that  $1barn = 10^{-24} \text{cm}^2$ .

If there is one particle in the initial state and we are studying decays we have to slightly modify our assumptions, because we assumed all particles to be stable in the previous discussion. However it turns out that the LSZ reduction formula also holds in this case. We only have to modify the initial state normalization  $\langle i|i\rangle = 2E_1V$  and find for the differential decay rate of a particle with energy  $E_1$ and 4-momentum  $p_1$ 

$$d\Gamma = \frac{|\mathcal{M}|^2}{2E_1} d\text{LIPS}_n(p_1) \tag{2.31}$$

and the decay rate is obtained by summing over all outgoing momenta and dividing by the symmetry factor  $S_F$ 

$$\Gamma = \frac{1}{S_F} \int d\Gamma .$$
(2.32)

Note that the decay rate is not a Lorenz scalar. In the centre-of-mass frame of the particle, there is  $E_1 = m_1$ , while the decay rate is smaller in any other frame by a factor  $\gamma = E_1/m_1$ , the relativistic boost factor, which accounts for the relativistic time dilation. Faster particles have an apparent longer

lifetime, e.g. muons generated in the atmosphere reach the Earth's surface due to this time dilation factor. For decays into two particles with masses  $m_1$  and  $m_2$  in the centre-of-mass frame we find

$$\Gamma = \frac{\lambda^{1/2} (M^2, m_1^2, m_2^2)}{32\pi M^3} \frac{1}{S_F} \int_{-1}^1 |\mathcal{M}|^2 d\cos\theta \,.$$
(2.33)

If there are multiple decay channels to different final state particle, the decay rates have to be summed

$$\Gamma_{tot} = \sum_{\text{all process}} \Gamma_i \tag{2.34}$$

For example charged pions usually decay to a muon and a neutrino

$$\pi^+ \to \mu^+ + \nu_\mu , \qquad (2.35)$$

but there are also rarer decay channels such as

$$\pi^+ \to e^+ + \nu_e \qquad \qquad \pi^+ \to \mu^+ + \nu_\mu + \gamma \qquad \qquad \pi^+ \to e^+ + \nu_e + \gamma . \tag{2.36}$$

## **3** A first attempt at relativistic quantum mechanics

In quantum mechanics nature is described by the wave function  $\psi(x,t)$ , whose time evolution is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi . \tag{3.1}$$

For a non-relativistic one-particle system, the Hamiltonian is generally given by

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) . \qquad (3.2)$$

In the absence of a potential V(x), the Schrödinger equation can be solved in terms of plane waves

$$Ne^{i\mathbf{p}\cdot\mathbf{x}-i\omega_{\mathbf{p}}t/\hbar}$$
 with  $\omega_{\mathbf{p}} = \frac{\hbar^2 \mathbf{p}^2}{2m}$  (3.3)

with some normalisation constant N.  $\omega_{\mathbf{p}}$  is the energy of the non-relativistic particle moving with momentum  $\hbar \mathbf{p}$ . The general solution is given by a superposition of the different solutions

$$\psi(t, \mathbf{x}) = \int d^3 p f(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x} - i\omega_p t/\hbar}$$
(3.4)

in terms of a function f subject to the normalisation  $\int |\psi|^2 d^3x = 1$ . The probability density of a free particle satisfies a *continuity equation* ( $\hbar = 1$ )

$$\frac{\partial}{\partial t}\psi^*\psi = \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi$$
(3.5)

$$\stackrel{(3.1)}{=} -i\psi^* \left(-\frac{\nabla^2}{2m}\psi\right) + i\left(-\frac{\nabla^2}{2m}\psi^*\right)\psi \tag{3.6}$$

$$=\frac{i}{2m}\left(\psi^*\nabla^2\psi - (\nabla^2\psi^*)\psi\right) \tag{3.7}$$

$$=\frac{i}{2m}\nabla\cdot\left(\psi^*\nabla\psi-(\nabla\psi^*)\psi\right) . \tag{3.8}$$

It is of the form of a continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0 \tag{3.9}$$

with the probability density  $\rho = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$  and the probability current

$$\mathbf{j} = \frac{1}{2im} \left( \psi^* \nabla \psi - (\nabla \psi^*) \psi \right) . \tag{3.10}$$

The continuity equation relates the probability to find the particle in a volume element  $\rho dV$  with the flow of the probability out of the volume element  $\mathbf{j} \cdot d\mathbf{S}$  via a surface  $d\mathbf{S}$ . The continuity equation ensures that the probability to find the particle anywhere in spacetime remains the same and it is thus an important ingredient in quantum mechanics.

For a single wave as defined in Eq. (3.3), the probability density is simply given by  $\rho = |N|^2$ , which can be interpreted as representing as the number density of particles, while the probability current is

$$\mathbf{j} = \frac{1}{2im} \left( \psi^* i \mathbf{p} \psi - (-i \mathbf{p} \psi^*) \psi \right) = \frac{\mathbf{p}}{2m} 2 \psi^* \psi = \frac{\mathbf{p}}{m} |N|^2 , \qquad (3.11)$$

which is proportional to the non-relativistic velocity  $\mathbf{p}/m$  of the particles. Thus the plane wave  $\psi(\mathbf{x}, t)$  represents a region of space with number density  $|N|^2$  particles per unit volume moving with average velocity  $\mathbf{p}/m$ .

The Schrödinger equation is inherently non-relativistic, since it treats time and position differently: while there is a second derivative with respect to x, there is only one derivative with respect to time. In order to obtain a Lorentz-invariant version of the Schrödinger equation we have to treat time and position on an equal footing, i.e. either use one derivative or a second derivative.

#### 3.1 Klein-Gordon equation

Let us start with second derivatives.

In special relativity a particle with 4-momentum  $p^{\mu}$  has to satisfy the relativistic dispersion relation

$$p^{\mu}p_{\mu} = E^2 - \mathbf{p}^2 = m^2 . \qquad (3.12)$$

Following the standard practice in quantum mechanics we replace the energy and momentum by operators (setting  $\hbar = 1$ )

$$E \to i\partial_t \qquad \mathbf{p} \to -i\nabla \qquad (3.13)$$

and postulate the wave equation for a relativistic spin-0 particle

$$m^{2}\phi = \left[ (i\partial_{t})^{2} - (-i\nabla)^{2} \right] \phi = (-\partial_{t}^{2} + \nabla^{2})\phi \qquad (\partial_{\mu}\partial^{\mu} + m^{2})\phi = 0 \qquad (3.14)$$

which is the so-called *Klein-Gordon equation*. By construction it is explicitly Lorentz-invariant.

The Klein-Gordon equation is a second-order partial differential equation and is solved in terms of plane waves

$$N \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega_k t) \qquad \text{with} \qquad \omega_k = \pm (\mathbf{k}^2 + m^2)^{1/2} . \tag{3.15}$$

Note that there are solutions with negative energy! This is a problem, because the Hilbert space is formed by all possible solutions.

A bigger problem is related to the probability density. Similarly to the non-relativistic Schrödinger equation, we can define a continuity equation. We start by considering the following construct

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2} \stackrel{(3.14)}{=} \phi^* \left( \nabla^2 \phi - m^2 \phi \right) - \phi \left( \nabla^2 \phi^* - m^2 \phi^* \right) = \phi^* \nabla^2 \phi - \phi \nabla^2 \phi^* \tag{3.16}$$

This can be rewritten in form of a continuity equation  $i\partial_t \rho + \nabla \cdot \mathbf{j} = 0$  with

$$\rho = -i\left(\phi^* \frac{\partial\phi}{\partial t} - \phi \frac{\partial\phi^*}{\partial t}\right) \qquad \qquad \mathbf{j} = i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right) , \qquad (3.17)$$

where we added -i to make the quantities real. Although the function  $\phi$  satisfies a continuity equation, we can not interpret it as a continuity equation for the probability density: For a plane wave we find that the density is given by

$$\rho = \omega_k \phi^* \phi - \phi(-\omega_k) \phi^* = 2\omega_k |N|^2$$
(3.18)

and thus proportional to the energy of the plane wave. As the density can become negative, it can not be interpreted as probability density.

#### 3.2 Dirac equation

As relativistic quantum mechanics with second derivatives has problems, let us try to use first derivatives. Dirac's strategy was to factor the energy-momentum relation into two factors linear in the 4-momentum

$$0 = p^{\mu} p_{\mu} - m^2 = (\beta^{\alpha} p_{\alpha} + m)(\gamma^{\lambda} p_{\lambda} - m) , \qquad (3.19)$$

where  $\beta^{\alpha}$  and  $\gamma^{\lambda}$ ,  $\alpha, \lambda = 0, 1, 2, 3$  are eight coefficients which have to be determined. Eq. (3.19) is solved if either of the two factors vanishes, i.e.

$$\beta^{\alpha} p_{\alpha} + m = 0$$
 or  $\gamma^{\lambda} p_{\lambda} - m = 0$ . (3.20)

We can use either of the two linear equations to define a relativistic equivalent of the Schödinger equation after the canonical substitution  $p_{\mu} \rightarrow i\partial_{\mu}$ . For example the second possibility with -m in Eq. (3.20) leads to

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi \tag{3.21}$$

$$\Rightarrow i\gamma^0 \partial_t \psi = \left[ -i\gamma^i \frac{\partial}{\partial x^i} + m \right] \psi \tag{3.22}$$

$$\Rightarrow i\partial_t \psi = \left[ -i(\gamma^0)^{-1} \gamma^i \frac{\partial}{\partial x^i} + m(\gamma^0)^{-1} \right] \psi \equiv \hat{H} \psi , \qquad (3.23)$$

where we used the definition of  $\partial_{\mu}$  in the second line and defined the Hamiltonian for the Dirac equation in the third line. So far we did not yet determine the coefficients  $\beta^{\alpha}$  and  $\gamma^{\lambda}$ . What are those coefficients? In order to answer this question we multiply out the right-hand side of Eq. (3.19)

$$(\beta^{\alpha}p_{\alpha}+m)(\gamma^{\lambda}p_{\lambda}-m) = \beta^{\alpha}p_{\alpha}\gamma^{\lambda}p_{\lambda} - m(\beta^{\lambda}-\gamma^{\lambda})p_{\lambda} - m^{2}.$$
(3.24)

A comparison with the left-hand side of Eq. (3.19) shows that  $\beta^{\lambda} = \gamma^{\lambda}$  and thus

$$p^{\mu}p_{\mu} = \gamma^{\alpha}p_{\alpha}\gamma^{\lambda}p_{\lambda} . \qquad (3.25)$$

Writing out Eq. (3.25) in components we find

$$(p^{0})^{2} - (p^{1})^{2} - (p^{2})^{2} - (p^{3})^{2} = (\gamma^{0}p_{0} - \gamma^{1}p_{1} - \gamma^{2}p_{2} - \gamma^{3}p_{3})^{2}$$

$$= (\gamma^{0})^{2}(p^{0})^{2} + (\gamma^{1})^{2}(p^{1})^{2} + (\gamma^{2})^{2}(p^{2})^{2} + (\gamma^{3})^{2}(p^{3})^{2}$$

$$(3.26)$$

$$= (\gamma^{0})^{2}(p^{0})^{2} + (\gamma^{1})^{2}(p^{1})^{2} + (\gamma^{2})^{2}(p^{2})^{2} + (\gamma^{3})^{2}(p^{3})^{2}$$

$$(3.26)$$

$$+ (\gamma^{0}\gamma^{1} + \gamma^{1}\gamma^{0})p^{0}p^{1} + (\gamma^{0}\gamma^{2} + \gamma^{2}\gamma^{0})p^{0}p^{2} + (\gamma^{0}\gamma^{3} + \gamma^{3}\gamma^{0})p^{0}p^{3} + (\gamma^{1}\gamma^{2} + \gamma^{2}\gamma^{1})p^{1}p^{2} + (\gamma^{2}\gamma^{3} + \gamma^{3}\gamma^{2})p^{2}p^{3} + (\gamma^{1}\gamma^{3} + \gamma^{3}\gamma^{1})p^{1}p^{3}$$

Comparing the left-hand side with the right-hand side, we find the following relations for the coefficients  $\gamma^{\lambda}$  for i = 1, 2, 3

$$(\gamma^0)^2 = 1$$
  $(\gamma^i)^2 = -1$   $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0$  (3.28)

The last condition can not be satisfied by ordinary commuting numbers (*c*-numbers), unless at least three coefficients vanish which is in contradiction to the other two conditions. The simplest way out is to assume that the coefficients are matrices. The conditions in Eq. (3.28) can be rewritten in a compact form

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} ,$$
 (3.29)

where  $\eta^{\mu\nu}$  is the (inverse) Minkowski metric and the curly braces are the *anticommutator*,

$$\{A, B\} \equiv AB + BA \tag{3.30}$$

In addition to the defining relation (3.29) for the  $\gamma$  matrices, we have to require that  $\gamma^0$  and  $\gamma^0 \gamma^i$  are hermitian matrices<sup>6</sup>, in order to have a hermitian Hamiltonian. Thus the matrices must be square matrices. This implies the relation

$$(\gamma^{\mu})^{\dagger} = \gamma^{0} \gamma^{\mu} \gamma^{0} \tag{3.31}$$

which is obvious for  $\mu = 0$  and it follows from the hermiticity of  $\gamma^0 \gamma^i$  for  $\mu = i$ 

$$\gamma^{0}\gamma^{i} = (\gamma^{0}\gamma^{i})^{\dagger} = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^{0} \Rightarrow (\gamma^{i})^{\dagger} = \gamma^{0}\gamma^{i}\gamma^{0} .$$
(3.32)

It turns out that the smallest matrices, which satisfy the above conditions are  $4 \times 4$  matrices. The set of matrices is not unique. One specific representation of the  $\gamma$  matrices is given by the *Dirac-Pauli* representation

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \qquad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix} \qquad (3.33)$$

where 1 denotes a 2 × 2 identity matrix, 0 a 2 × 2 matrix of zeros and  $\sigma^i$ , i = 1, 2, 3 are the *Pauli* matrices

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.34)$$

Another common representation for the  $\gamma$  matrices is the Weyl or chiral representation

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \tag{3.35}$$

where  $\sigma^0$  is the 2 × 2 identity matrix,  $\sigma^i$  the Pauli matrices, and  $\bar{\sigma}^0 = \sigma^0$  and  $\bar{\sigma}^i = -\sigma^i$ . In the following we will use the Pauli-Dirac representation, unless explicitly specified.

Summarising the above, we find that

$$p^{\mu}p_{\mu} - m^2 = \left(\gamma^{\alpha}p_{\alpha} + m\right)\left(\gamma^{\lambda}p_{\lambda} - m\right) = 0.$$
(3.36)

<sup>6</sup>Note that  $(\gamma^0)^{-1} = \gamma^0$  from Eq. (3.28).

Either factor can be used as relativistic wave equation. Conventionally the second factor (with -m) is picked. After the substitution  $p_{\mu} \rightarrow i\partial_{\mu}$  we obtain the *Dirac equation* 

$$(i\partial - m)\psi = (i\gamma^{\mu}\partial_{\mu} - m)\psi = 0, \qquad (3.37)$$

where we introduced the commonly-used Feynman-slash notation  $\mathscr{J} \equiv \gamma^{\mu} \partial_{\mu}$ . Note that the wave function  $\psi$  is not a number, but has four components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} . \tag{3.38}$$

It is called a *Dirac spinor* and should *not* be confused with a Lorentz 4-vector, since it does not transform as a 4-vector under Lorentz transformations, but as a spinor. The individual components are *c*-numbers. You find details in any book on relativistic quantum mechanics (e.g. Schwabl: "Advanced Quantum Mechanics"). Furthermore any solution to the Dirac equation is also a solution to the Klein-Gordon equation (3.14). If we multiply the Dirac equation with  $(i\gamma^{\nu}\partial_{\nu} + m)$  from the left we obtain

$$0 = (i\gamma^{\nu}\partial_{\nu} + m)(i\gamma^{\mu}\partial_{\mu} - m)\psi$$
(3.39)

$$= \left(-\gamma^{\nu}\gamma^{\mu}\partial_{\nu}\partial_{\mu} + m(i\gamma^{\mu}\partial_{\mu} - i\gamma^{\nu}\partial_{\nu}) - m^{2}\right)\psi$$
(3.40)

$$= \left(-\frac{1}{2}\left\{\gamma^{\nu},\gamma^{\mu}\right\}\partial_{\nu}\partial_{\mu} - m^{2}\right)\psi$$
(3.41)

$$\stackrel{(3.29)}{=} \left(-\partial^{\mu}\partial_{\mu} - m^{2}\right)\psi\tag{3.42}$$

$$= -\left(\partial^{\mu}\partial_{\mu} + m^{2}\right)\psi. \qquad (3.43)$$

Similarly to the Klein-Gordon equation, we can define a continuity equation. For this we need the Dirac equation for the (complex) conjugate Dirac spinor

$$0 = (i\gamma^{\mu}\partial_{\mu}\psi - m\psi)^{\dagger} = -i\partial_{\mu}\psi^{\dagger}\gamma^{\mu\dagger} - m\psi^{\dagger} \stackrel{(3.31)}{=} -i\partial_{\mu}\psi^{\dagger}\gamma^{0}\gamma^{\mu}\gamma^{0} - m\psi^{\dagger} .$$
(3.44)

This motivates the definition of the *adjoint spinor* 

$$\bar{\psi} = \psi^{\dagger} \gamma^0 \tag{3.45}$$

which satisfies the equation

$$0 = -i\partial_{\mu}\bar{\psi}\gamma^{\mu} - m\bar{\psi} \tag{3.46}$$

Using the above equation and the Dirac equation we find

$$\partial_{\mu} \left( \bar{\psi} \gamma^{\mu} \psi \right) = (\partial_{\mu} \bar{\psi}) \gamma^{\mu} \psi + \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi \overset{(3.46)}{\stackrel{=}{=}} i m \bar{\psi} \psi - i \bar{\psi} m \psi = 0 \tag{3.47}$$

and thus we find the density  $\rho \equiv \bar{\psi}\gamma^0\psi$  and the current  $\mathbf{j} = \bar{\psi}\gamma^i\psi$  of the continuity equation. The density  $\rho$  is positive definite

$$\rho = \bar{\psi}\gamma^0\psi = \psi^{\dagger}\gamma^0\gamma^0\psi = \psi^{\dagger}\psi = \sum_i |\psi_i|^2 \ge 0$$
(3.48)

and thus can be interpreted as a probability density in contrast to the Klein-Gordon equation. It can be shown that  $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$  transforms as a 4-vector under Lorentz transformations.

#### 3.2.1 Solutions of the Dirac equation

We expect that the solutions of the Dirac equation are also given by plane waves, i.e. they are of the form

$$\psi(x^{\mu}) = u(p^{\mu})e^{-ip_{\mu}x^{\mu}} \tag{3.49}$$

where  $u(p^{\mu})$  is a time-independent 4-component spinor. Substitution into the Dirac equation (3.37) shows

$$0 = (i\gamma^{\mu}\partial_{\mu} - m)\psi = (\gamma^{0}p_{0} - \sum_{i}\gamma^{i}p_{i} - m)u(p^{\mu})e^{-ip_{\mu}x^{\mu}}.$$
(3.50)

Thus Dirac equation simplifies to an algebraic problem and the solutions to the Dirac equation have to satisfy

$$0 = (\gamma^{\mu} p_{\mu} - m)u . (3.51)$$

We first solve it for a particle at rest with  $\mathbf{p} = 0$  and energy  $E = \pm m$ . In this case Eq. (3.51) simplifies to

$$0 = (\gamma^{0}E - m)u = \begin{pmatrix} E - m & & \\ & E - m & \\ & & -E - m \\ & & & -E - m \end{pmatrix} u$$
(3.52)

which has four independent solutions

$$u_{1}(E = m, \mathbf{p} = 0) = N \begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} \qquad u_{2}(E = m, \mathbf{p} = 0) = N \begin{pmatrix} 0\\ 1\\ 0\\ 0 \end{pmatrix} \qquad (3.53)$$
$$u_{3}(E = -m, \mathbf{p} = 0) = N \begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 0 \end{pmatrix} \qquad u_{4}(E = -m, \mathbf{p} = 0) = N \begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 1 \end{pmatrix} \qquad (3.54)$$

with a normalisation constant N, which we will determine later. The first two  $(u_{1,2})$  are for positive energies E = +m and the last two  $(u_{3,4})$  for negative energies E = -m. The corresponding wave functions for a particle at rest are given by

$$\psi_1 = u_1 e^{-imt}$$
  $\psi_2 = u_2 e^{-imt}$   $\psi_3 = u_3 e^{+imt}$   $\psi_4 = u_4 e^{+imt}$  (3.55)

For general  $\mathbf{p} \neq 0$ , the equation for *u* becomes

$$\begin{bmatrix} \gamma^0 E - \begin{pmatrix} 0 & \sigma \cdot \mathbf{p} \\ -\sigma \cdot \mathbf{p} & 0 \end{pmatrix} - m \end{bmatrix} u = 0$$
(3.56)

The solutions to this equation are given by

$$u_{1}(E > 0) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_{z}}{E+m} \\ \frac{p_{x}+ip_{y}}{E+m} \end{pmatrix} \qquad u_{2}(E > 0) = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_{x}-ip_{y}}{E+m} \\ \frac{-p_{z}}{E+m} \end{pmatrix} \qquad (3.57)$$
$$u_{3}(E < 0) = N \begin{pmatrix} \frac{p_{z}}{E-m} \\ \frac{p_{x}+ip_{y}}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_{4}(E < 0) = N \begin{pmatrix} \frac{p_{x}-ip_{y}}{E-m} \\ \frac{-p_{z}}{E-m} \\ 0 \\ 1 \end{pmatrix} \qquad (3.58)$$

Like for the Klein-Gordon equation, there are negative energy solutions which we have to be interpreted.

#### 3.2.2 Antiparticles

The presence of negative energy solutions seems unavoidable and can not be simply discarded. If negative energy solutions represent negative energy particle states, positive energy particle states should fall into the lower negative energy particle states. Dirac proposed that the vacuum corresponds to the case, where all negative energy eigenstates are filled, the so-called *Dirac sea*, and thus due to the Pauli exclusion principle the positive energy particle states won't fall into the occupied negative energy states.

The modern interpretation is due to Feynman and Stückelberg. The E < 0 states are interpreted as negative energy particle which propagate backwards in time. These negative energy particles correspond to physical positive energy antiparticle states with opposite charge, which propagate forwards in time. As the time dependence of the wave function  $e^{-iEt}$  is unchanged under the simultaneous transformation  $(E, t) \rightarrow (-E, -t)$ , the two pictures are mathematically equivalent.

As  $u_3$  and  $u_4$  are interpreted as travelling backwards in time, the momentum in the spinor is the negative of the physical momentum. It is easier to work with antiparticle spinors which we define by reversing the signs of  $(E, \mathbf{p})$ 

$$v_1(|E|, \mathbf{p})e^{i(|E|t - \mathbf{p} \cdot \mathbf{x})} \equiv u_4(-|E|, -\mathbf{p})e^{-i((-|E|)t - (-\mathbf{p}) \cdot \mathbf{x})}$$
(3.59)

$$v_2(|E|, \mathbf{p})e^{i(|E|t - \mathbf{p} \cdot \mathbf{x})} \equiv u_3(-|E|, -\mathbf{p})e^{-i((-|E|)t - (-\mathbf{p}) \cdot \mathbf{x})}$$
(3.60)

and thus

$$v_{1} = N \begin{pmatrix} \frac{p_{x} - ip_{y}}{E+m} \\ -\frac{p_{z}}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad \qquad v_{2} = N \begin{pmatrix} \frac{p_{z}}{E+m} \\ \frac{p_{x} + ip_{y}}{E+m} \\ 1 \\ 0 \end{pmatrix}$$
(3.61)

and the wave functions are given by

$$\psi_i = v_i e^{-i(\mathbf{p} \cdot \mathbf{x} - Et)} . \tag{3.62}$$

If we substitute it into the Dirac equation we obtain

$$(\gamma^{\mu}p_{\mu} + m)v = 0 \tag{3.63}$$

We did not fix the normalisation yet. There are different conventions. A common one is to normalise the density to

$$\rho = \psi^{\dagger}\psi = 2E . \tag{3.64}$$

For example for the  $\psi_1$  solution we obtain explicitly

$$\psi_1^{\dagger}\psi_1 = u_1^{\dagger}u_1 = |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2}\right) = |N|^2 \frac{2E}{E+m} \qquad \Rightarrow \qquad N = \sqrt{E+m} \quad (3.65)$$

The same normalisation is obtained for all u and v spinors.

The full solution is a linear combination of the 4 linearly-independent  $\psi_i$ , summed over waves of all possible momenta

$$\psi(x^{\mu}) = \sum_{i=1,2} \int \frac{d^3 p}{(2\pi)^3} \left[ A_i(\mathbf{p}) u_i(E, \mathbf{p}) e^{-ip_{\mu}x^{\mu}} + B_i(\mathbf{p}) v_i(E, \mathbf{p}) e^{ip_{\mu}x^{\mu}} \right] .$$
(3.66)

where  $A_i$  and  $B_i$  are c-number valued functions determined by the initial conditions.

#### 3.2.3 Spin, helicity, and chirality

For a particle at rest, particles are described by the top two components of the Dirac spinor and antiparticles by the lower two components of the Dirac spinor. In order to interpret the different components of the spinors, we have to introduce one more concept: the spin operator

$$\hat{S}_3 = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0\\ 0 & \sigma_3 \end{pmatrix} \tag{3.67}$$

with the Pauli spin matrix  $\sigma_3$ . The spinors  $u_1$  and  $u_2$  are eigenstates of the spin operator with eigenvalues +1/2 and -1/2 respectively. Thus at rest,  $u_1$  describes a particle with spin up and  $u_2$ describes a particle with spin down. We have to be more careful, when considering the antiparticle spinors. Applying the Hamiltonian and momentum operator to  $\psi = v(E, \mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}$  does not give the physical momentum

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t} = -E\psi \qquad \qquad \hat{\mathbf{p}}\psi = -\mathbf{p}\psi \qquad (3.68)$$

The physical energy and momentum are obtained via the operators

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \qquad \qquad \hat{\mathbf{p}}^{(v)} = i\nabla \qquad (3.69)$$

Also the operator for the orbital angular momentum has to be adjusted  $\mathbf{L} = \mathbf{r} \times \mathbf{p} \to \mathbf{L}^{(v)} = \mathbf{r} \times \mathbf{p}^{(v)}$ . Finally, in order for the commutator

$$[\hat{H}, (\hat{\mathbf{L}} + \hat{\mathbf{S}})^2] = 0 \tag{3.70}$$

to vanish, i.e. the sum of orbital angular momentum and spin is conserved, also the spin operator has to change sign for antiparticles

$$\hat{S}^{(v)} = -\hat{S} \ . \tag{3.71}$$

Thus at rest the spinor  $v_1$  describes an antiparticle with spin up and the spinor  $v_2$  describes an antiparticle with spin down.

The spin operator  $\hat{S}_3$  commutes with the Hamiltonian  $\hat{H}$  with  $\mathbf{p} = 0$ , but not for  $\mathbf{p} \neq 0$  and thus is not a good quantum number for  $\mathbf{p} \neq 0$ . However we can define an operator which commutes for  $\mathbf{p} \neq 0$ , the *helicity* operator

$$h \equiv \frac{1}{2|\mathbf{p}|} \begin{pmatrix} \sigma \cdot \mathbf{p} & 0\\ 0 & \sigma \cdot \mathbf{p} \end{pmatrix} .$$
(3.72)

It is the projection of spin onto the momentum. For  $\mathbf{p} = (0, 0, p_z)$ , it exactly agrees with the spin operator for a particle at rest. It can be shown that it commutes with the Hamiltonian  $[\hat{H}, h] = 0$ and thus is a conserved quantity. For  $p_{x,y} = 0$ , the  $u_1$  ( $u_2$ ) spinor describes a particle with positive (negative) helicity and  $v_1$  ( $v_2$ ) describes an antiparticle with negative (positive) helicity. Similarly to spin, we have to change the sign for antiparticles. A general helicity eigenstate can be constructed by searching for simultaneous eigenstates of  $\hat{H}$  and h. Helicity agrees with spin for  $\mathbf{p} \to (0, 0, p_z)$ :  $\hat{h} \to \hat{S}_3$ . In the ultra-relativistic limit,  $E \gg m$  the helicity eigenstates take the form

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} \qquad u_{\downarrow} = \sqrt{E} \begin{pmatrix} \xi_2 \\ -\xi_2 \end{pmatrix} \qquad v_{\downarrow} = \sqrt{E} \begin{pmatrix} -\xi_2 \\ \xi_2 \end{pmatrix} \qquad v_{\uparrow} = \sqrt{E} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} , \qquad (3.73)$$

where the 2-component spinors  $\xi_i$  are eigenstates of  $\frac{\sigma \cdot \mathbf{p}}{2|\mathbf{p}|}$ . In polar coordinates, where the 3-momentum takes the form  $\mathbf{p} = p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$ , they are given by

$$\xi_1 = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \qquad \qquad \xi_2 = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} . \tag{3.74}$$

See Thompson for details.

Helicity commutes with the Hamiltonian and is conserved, but it is not invariant under Lorentz transformations for massive particles, because the direction (sign) of the 3-momentum  $\mathbf{p}$  is not Lorentz invariant.

Another commonly used concept is *chirality* which is Lorentz invariant, but not conserved: We can form another  $\gamma$ -matrix out of the 4  $\gamma^{\mu}$ , the so-called  $\gamma_5$  matrix:

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.75}$$

In the Pauli-Dirac representation it is given by

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.76}$$

The  $\gamma_5$  matrix satisfies the following properties

$$(\gamma_5)^2 = 1$$
  $\gamma_5^{\dagger} = \gamma_5$   $\{\gamma_5, \gamma^{\mu}\} = 0$  (3.77)

which can be demonstrated by a straightforward calculation. In the ultrarelativistic limit  $E \gg m$ , the helicity eigenstates are also eigenstates of the  $\gamma_5$  matrix (chirality operator), in particular for  $p_{x,y} = 0$  they also agree with the spin eigenstates and thus

$$\gamma_5 u_1 = u_1 \qquad \gamma_5 u_2 = -u_2 \tag{3.78}$$

$$\gamma_5 v_1 = -v_1 \qquad \gamma_5 v_2 = v_2 . \tag{3.79}$$

However in general the spin, helicity and chirality do not coincide. We can define projection operators onto the right-chiral  $P_R = (1 + \gamma_5)/2$  and left-chiral  $P_L = (1 - \gamma_5)/2$  particle states. They satisfy the usual properties of a projection operator  $P_{R,L}^2 = P_{R,L}$  and divide the space in two chiralities, i.e.  $P_L + P_R = 1$  and  $P_L P_R = 0$ . In particular we find for the scalar  $\bar{\psi}\psi$  and vector bilinears  $\bar{\psi}\gamma^{\mu}\psi$ 

$$\bar{\psi}\psi = \bar{\psi}(P_L + P_R)\psi = \bar{\psi}(P_L^2 + P_R^2)\psi = (P_R\psi)^{\dagger}\gamma^0 P_L\psi + (P_L\psi)^{\dagger}\gamma^0 P_R\psi$$
(3.80)

$$\bar{\psi}\gamma^{\mu}\psi = \bar{\psi}\gamma^{\mu}(P_L + P_R)\psi = \bar{\psi}\gamma^{\mu}(P_L^2 + P_R^2)\psi = (P_L\psi)^{\dagger}\gamma^0\gamma^{\mu}P_L\psi + (P_R\psi)^{\dagger}\gamma^0\gamma^{\mu}P_R\psi$$
(3.81)

Thus the conserved current  $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$  splits into a sum of left- and right-chiral parts. Furthermore we will later see that the mass term in the Dirac Lagrangian  $m\bar{\psi}\psi$  connects different chiralities. For massless particles the Dirac equation splits into one equation for left-chiral particles and one for rightchiral particles. The chiral or Weyl representation is specially suited to discuss *chiral theories* like the Standard Model of particle physics, where left-chiral and right-chiral particles have different quantum numbers. In the Weyl representation  $\gamma_5$  is given by

$$\gamma_5 = \begin{pmatrix} -1 & 0\\ 0 & 1 \end{pmatrix} \,. \tag{3.82}$$

#### 3.2.4 Parity

*Parity* is defined as

$$x^{\mu} = (t, \mathbf{x}) \to x'^{\mu} = (t, -\mathbf{x})$$
 (3.83)

Its action on Lorentz 4-vectors  $x^{\mu} \to x'^{\mu} = (Px)^{\mu} = P^{\mu}_{\nu} x^{\nu}$  can be described by the matrix

$$(P^{\mu}_{\nu}) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \\ & & & -1 \end{pmatrix} .$$
 (3.84)

Applying parity twice maps the 4-vector back onto itself, i.e.  $P^2 = 1$ .

We can similarly define a parity operator for Dirac spinors  $\psi(x) \to \psi'(x') = \hat{P}\psi(x)$ , where x' = Px. In order to determine its form, we consider the Dirac equation in the primed (parity-transformed) system

$$0 = (i\gamma^0 \partial_{t'} + i\gamma^i \nabla'_i - m)\psi'(x')$$
(3.85)

and transform it back to the unprimed system using the parity transformation. We first use  $\psi'(x') = \hat{P}\psi(x)$  and obtain

$$0 = (i\gamma^0 \partial_{t'} + i\gamma^i \nabla'_i - m) \hat{P} \psi(x) . \qquad (3.86)$$

As the parity operator  $\hat{P}$  acts in spinors space, but does not depend on the coordinates  $x^{\mu}$ , we find

$$0 = (i\gamma^0 \hat{P}\partial_{t'} + i\gamma^i \hat{P}\nabla'_i - \hat{P}m)\psi(x) . \qquad (3.87)$$

Next we use change the derivatives from derivatives with respect to the primed coordinates to the ones of the unprimed coordinates as defined in Eq. (3.83) which implies  $\frac{\partial x^{\mu}}{\partial x'^{\nu}} = P^{\mu}_{\ \nu}$ 

$$0 = (i\gamma^0 \hat{P}\partial_t - i\gamma^i \hat{P}\nabla_i - \hat{P}m)\psi(x) .$$
(3.88)

Finally, we multiply the Dirac equation by  $\hat{P}$  from the left and find

$$0 = (i\hat{P}\gamma^0\hat{P}\partial_t - i\hat{P}\gamma^i\hat{P}\nabla_i - m)\psi(x) .$$
(3.89)

As  $\psi$  is a solution to the usual Dirac equation  $(i\partial - m)\psi(x)$ , we can compare the coefficients of the derivatives and find the following conditions for  $\hat{P}$ :  $\hat{P}\gamma^0\hat{P} = \gamma^0$  and  $\hat{P}\gamma^i\hat{P} = -\gamma^i$  which can be rewritten as

$$\hat{P}\gamma^0 = \gamma^0 \hat{P} \qquad \qquad \hat{P}\gamma^i + \gamma^i \hat{P} = 0 \qquad (3.90)$$

A comparison with the anticommutation relation  $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$  implies  $\hat{P} = \pm \gamma^{0}$ . The common choice is

$$\hat{P} = \gamma^0 \tag{3.91}$$

The *intrinsic* parity of a fundamental particle is defined by the action of the parity operator  $\hat{P} = \gamma^0$ on a spinor for a particle at rest. We find

$$\hat{P}u_i(m,0) = u_i(m,0)$$
  $\hat{P}v_i(m,0) = -v_i(m,0)$ , (3.92)

i.e. the intrinsic parity of a spin  $\frac{1}{2}$  particle is opposite from the one of the antiparticle.

If we had chosen  $\hat{P} = -\gamma^0$ , the intrinsic parities of particle and antiparticle would have been reversed. This is without any physical significance, because particles and antiparticles are always created and destroyed in pairs.

For a particle with 3-momentum  $\mathbf{p}$ , the action of the parity operator is

$$\hat{P}u_i(E,\mathbf{p}) = +u_i(E,-\mathbf{p}) \qquad \qquad \hat{P}v_i(E,\mathbf{p}) = -v_i(E,-\mathbf{p}) \qquad (3.93)$$

Note that the action of parity changes 3-momentum, but not spin.

## 4 Interactions

#### 4.1 Lippmann-Schwinger equation

Consider a Hamiltonian H which consists of two parts, a free Hamiltonian  $H_0$  which can be exactly solved, and an interaction part V, which gives a small correction

$$H = H_0 + V$$
 . (4.1)

In contrast to many problems in quantum mechanics, we are typically working with free-particle states with a continuous energy spectrum. We are interested in solutions with the same energy E for both the free Hamiltonian  $H_0$  and the full Hamiltonian H: Let  $|\phi\rangle$  be an eigenstate of  $H_0$  with energy E

$$H_0 \left| \phi \right\rangle = E \left| \phi \right\rangle \tag{4.2}$$

and  $|\psi\rangle$  is an eigenstate of H with energy E

$$H\left|\psi\right\rangle = E\left|\psi\right\rangle \tag{4.3}$$

In the limit  $V \to 0$ ,  $|\psi\rangle \to |\phi\rangle$ . We are specially interested in scattering. It is possible to formally write the full wave function  $|\psi\rangle$  as the sum of the free wave function  $|\phi\rangle$  and a scattering term

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} V |\psi\rangle , \qquad (4.4)$$

which can be directly verified by multiplying the equation by  $E-H_0$ . The equation is called *Lippmann-Schwinger equation*. In the limit  $V \rightarrow 0$ , the second term in the Lippmann-Schwinger equation vanishes and we are left with the wave function of the free particle. The form is useful for a scattering calculation, where at early and late times the particles are free (non-interacting), while the potential V acts at intermediate times.

Formally, the inverse of  $E - H_0$  is not defined, since E is an eigenvalue of  $H_0$ . In order to regulate the expression, we add a small imaginary part and define the *Lippmann-Schwinger kernel* 

$$\Pi_{LS}^{\pm} = \frac{1}{E - H_0 \pm i\epsilon} \tag{4.5}$$

with the understanding that we take  $\epsilon \to 0^+$  at the end of the calculation. In the following discussion we will not explicitly write this small imaginary part. Using the Lippmann-Schwinger kernel, the Lippmann-Schwinger equation takes the form

$$|\psi\rangle = |\phi\rangle + \Pi_{LS} V |\psi\rangle . \tag{4.6}$$

The left-hand side of the Lippmann-Schwinger equation depends on the solution  $|\psi\rangle$ . In order to solve it we introduce the transfer matrix T by

$$T \left| \phi \right\rangle \equiv V \left| \psi \right\rangle \tag{4.7}$$

which leads to

$$|\psi\rangle = (1 + \Pi_{LS}T) |\phi\rangle . \tag{4.8}$$

We multiply the equation by the potential V and obtain

$$T |\phi\rangle = (V + V\Pi_{LS}T) |\phi\rangle . \tag{4.9}$$

As the equation holds for any state  $|\phi\rangle$ , we obtain an operator equation for T

$$T = V + V\Pi_{LS}T \tag{4.10}$$

which can be solved iteratively.

$$T = V + V\Pi_{LS}V + V\Pi_{LS}V\Pi_{LS}V + \dots$$
(4.11)

In the following we will be only working with eigenstates of the free Hamiltonian  $H_0$ , i.e. the basis of  $|\phi\rangle$  states. Sandwiching the operator equation between initial and final states  $|i\rangle$  and  $|f\rangle$  and inserting the complete set of states  $\sum_{i} |j\rangle \langle j|$ 

$$\langle f|T|i\rangle = \langle f|V|i\rangle + \langle f|V\sum_{j}|j\rangle \langle j|\Pi_{LS}V|i\rangle + \dots , \qquad (4.12)$$

the matrix element  $T_{fi} = \langle f | T | i \rangle$  can be expressed as

$$T_{fi} = V_{fi} + V_{fj} \Pi_{LS}(j) V_{ji} + V_{fj} \Pi_{LS}(j) V_{jk} \Pi_{LS}(k) V_{ki} + \dots$$
(4.13)

where we introduced the matrix elements  $V_{jk} = \langle j | V | k \rangle$  and  $\Pi_{LS}(k) = \frac{1}{E - E_k}$ .

Let us dwell on the final result. It describes the transition from an initial state  $|i\rangle$  at an early time, which describes a free and non-interacting particle, to a final state  $|f\rangle$  at late time which is also free and non-interacting. At intermediate times, a number of interactions occur which are described by the potential  $V_{jk}$ . Each insertion of the potential projects the state  $|k\rangle$  onto a state  $|j\rangle$ . In between the interactions, the propagation of the state is described by the Lippmann-Schwinger kernel  $\Pi_{LS}$ . The transfer matrix element is then given by the sum over the possible number of interactions V. The first term  $V_{fi}$  gives the Born approximation (or first Born approximation), the second term the second Born approximation, etc. This series can be nicely represented in a diagrammatic way using so-called Feynman diagrams.

#### 4.2 Time-ordered perturbation theory

Consider a particle interaction  $a + b \rightarrow c + d$ , which occurs via an intermediate state with exchanged particles X and  $\tilde{X}$ . You can think of the scattering  $e^-\nu_{\mu} \rightarrow \nu_e \mu^-$  which is mediated by  $W^{\pm}$  bosons. Fig. 6 illustrates the possible contributions to this scattering process. The time evolution in both diagrams is from left to right. The diagram on the left-hand side describes the emission of X from a, i.e. there is an intermediate state b + c + X, and the absorption of X by b at a later time. Similarly



Figure 6: Two possible time-orderings for the process  $a + b \rightarrow c + d$ . The labels i, j, f indicate initial, intermediate and final state. Time increases from left to right and the vertical direction indicates spatial separation.

the diagram on the right-hand side describes the emission of  $\tilde{X}$  by b leading to the intermediate state  $d + \tilde{X} + a$  and the absorption of  $\tilde{X}$  by a.

Hence to leading order there are two contributions to the transfer matrix  $T_{fi}$  as shown in Eq. (4.13). The first contribution is

$$T_{fi,1} = V_{fj}\Pi_{LS}(j)V_{ji} = \frac{\langle d|V|b+X\rangle\langle c+X|V|a\rangle}{(E_a+E_b)-(E_c+E_X+E_b)} = \frac{\langle d|V|b+X\rangle\langle c+X|V|a\rangle}{E_a-E_c-E_X} , \qquad (4.14)$$

where we used the energies of the initial and intermediate state. Note that the energy of the intermediate state is not the same as the initial state,  $E_i \neq E_j$ . This is allowed by the Heisenberg uncertainty relation for a short period of time. The 3-momentum is conserved at each vertex. The interaction at the two vertices is defined by the non-invariant matrix elements  $V_{fj}$  and  $V_{ji}$ . They are related to the Lorentz-invariant matrix element<sup>7</sup>

$$V_{ji} = \mathcal{M}_{ji} \prod_{k} \frac{1}{\sqrt{2E_k}} \tag{4.15}$$

where k runs over all particles which take part in the interaction. The two relevant interaction potentials are thus given by

$$V_{ji} = \langle c + X | V | a \rangle = \frac{\mathcal{M}_{a \to c + X}}{\sqrt{2E_a 2E_c 2E_X}} \qquad V_{fj} = \langle d | V | b + X \rangle = \frac{\mathcal{M}_{b + X \to d}}{\sqrt{2E_b 2E_d 2E_X}}$$
(4.16)

The simplest Lorentz-invariant matrix element is a scalar. Let us assume  $\mathcal{M}_{a\to c+X} = g_a$  and  $\mathcal{M}_{b+X\to d} = g_b$ . Similarly we express the transfer matrix element in terms of Lorentz-invariant matrix element  $\mathcal{M}_{fi,1}$ 

$$\mathcal{M}_{fi,1} = \sqrt{2E_a 2E_b 2E_c 2E_d} T_{fi,1} \tag{4.17}$$

$$= \frac{g_b}{\sqrt{2E_b 2E_d 2E_X}} \frac{g_a}{\sqrt{2E_a 2E_c 2E_X}} \frac{1}{E_a - E_c - E_X}$$
(4.18)

$$=\frac{1}{2E_X}\frac{g_a g_b}{E_a - E_c - E_X}$$
(4.19)

We can similarly calculate the Lorentz-invariant matrix element for the second diagram, where we assume that  $\tilde{X}$  and X have the same mass but opposite charge and obtain

$$\mathcal{M}_{fi,2} = \frac{1}{2E_X} \frac{g_a g_b}{E_b - E_d - E_X}$$
(4.20)

<sup>&</sup>lt;sup>7</sup>This is due to the normalisation of the wave function  $\psi^{\dagger}\psi = 2E$ .

for the second diagram. The two different amplitudes interfere and the sum of the two time-ordered amplitudes is

$$\mathcal{M}_{fi} = \mathcal{M}_{fi,1} + \mathcal{M}_{fi,2} \tag{4.21}$$

$$=\frac{g_a g_b}{2E_X} \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X}\right)$$
(4.22)

$$=\frac{g_a g_b}{2E_X} \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X}\right)$$
(4.23)

$$=\frac{g_a g_b}{(E_a - E_c)^2 - E_X^2} \tag{4.24}$$

where we used overall energy conservation  $E_a + E_b = E_c + E_d$  in the second-to-last line. The energy of the intermediate particle is related to its 3-momentum via the dispersion relation  $E_X^2 = \mathbf{p}_X^2 + m_X^2$ which can be rewritten using 3-momentum conservation at each vertex as  $E_X^2 = (\mathbf{p}_a - \mathbf{p}_c)^2 + m_X^2$  and thus the transfer matrix element can be expressed in terms of the 4-momentum

$$q = p_a - p_c \tag{4.25}$$

of the exchanged particle (in the first diagram)

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\mathbf{p}_a - \mathbf{p}_c)^2 - m_X^2} = \frac{g_a g_b}{q^2 - m_X^2} , \qquad (4.26)$$

where  $g_a$  and  $g_b$  parameterise the interaction vertices and the term

$$\frac{1}{q^2 - m_X^2} \tag{4.27}$$

is the propagator, which describes the exchanged particle. Note that  $q^2 \neq m_X^2$ , i.e. the X particle is not on-shell, but a virtual particle.

#### 4.3 Feynman diagrams

In QFT, the sum over all time-orderings is represented by a *Feynman diagram*. The Feynman diagram for the process discussed in the previous section is



where the right-hand side shows the two different time-orderings from Fig. 6. The line of the virtual particle X represents summing over all time-ordered diagrams on the right-hand side. From the Feynman diagram, we can directly read off the corresponding Lorentz-invariant matrix element using the so-called *Feynman rules*. In the previous section we saw an example, where all particles have



Figure 7: Loop diagrams

spin 0. In this case, the vertex factors  $g_{a,b}$  are constants and the propagator of the intermediate state particle is  $(q^2 - m_X^2)^{-1}$ . The Feynman rules for theories with fermions and gauge bosons are more complicated.

Feynman diagrams are a calculational tool to represent the different contributions to particle physics processes. In QFT, each diagram represents one term in the *Dyson series*, a perturbative series obtained from time-dependent perturbation theory. Equivalently, we can also discuss it in terms of time-ordered perturbation theory based on Eq. (4.13). In the previous example, we discussed the lowest order contribution, which required two vertices. Higher order corrections are described by terms with more powers of the potential V. Examples of diagrams for higher-order terms are shown in Fig. 7. These diagrams are loop diagrams. The lowest order diagram, which we calculated in the previous lecture is a tree-level diagram.

Once we have all Feynman diagrams which contribute to one process, there is a set of rules that allows to compute the corresponding term in the perturbative expansion. For example in the previous example, we found

$$a \qquad g_a \qquad p_c \qquad g_b \qquad p_c \qquad g_b \qquad g_b \qquad M = g_a \times \frac{1}{q^2 - m_X^2} \times g_b \qquad (4.29)$$

Let us consider the case, where  $g_a = g_b \equiv g$ . Then we find for the first one-loop diagram in Fig. 7

where we inserted one factor of g for each vertex, one propagator for each (internal) line connecting two vertices. As not all four 4-momenta are fixed by energy momentum conservation at each vertex

$$q_{X_1} = p_a - p_c \qquad q_{X_2} = p_d - p_b \qquad q_{k_2} = k_1 - q_{X_1} \qquad (4.31)$$

in terms of the external momenta  $p_{a,b,c,d}$  and other 4-momenta, we integrate over the remaining 4momentum  $k_1$ . As the 4-momentum is conserved overall,  $p_a + p_b = p_c + p_d$ , the 4-momenta of the two X particles are the same  $q_{X_1} = q_{X_2} = p_a - p_c$ .

#### 4.4 ABC theory

Consider a theory of spin-0 particles A, B, and C with masses  $m_A$ ,  $m_B$  and  $m_C$ , respectively, and one interaction vertex



where g is the coupling constant. Using the Feynman rule for the vertex and the propagators, we can calculate the scattering amplitude for the process

$$A(p_1)A(p_2) \to B(p_3)B(p_4)$$
 (4.33)

where  $p_{1,2,3,4}$  denotes the 4-momenta of the particles. There are two possible Feynman diagrams

where the final state particles are exchanged in the second diagram. There are two distinct Feynman diagrams, because there are two identical particles in the final state and we can not observe which one emerges from which vertex. Following the Feynman rules we obtain for the scattering amplitude of the first diagram

$$\mathcal{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2} = \frac{g^2}{t - m_C^2}$$
(4.35)

where we used energy-momentum conservation at the top vertex to express q in terms of  $p_1$  and  $p_3$ and introduced the Mandelstam variable  $t = (p_1 - p_3)^2$ . Similarly we find for the second diagram

$$\mathcal{M}_2 = \frac{g^2}{(p_1 - p_4)^2 - m_C^2} = \frac{g^2}{u - m_C^2}$$
(4.36)

where we introduced the Mandelstam variable  $u = (p_1 - p_4)^2$ . The total matrix element is given by the sum of the two expressions

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2} \,. \tag{4.37}$$

#### 4.5 Yukawa potential: non-relativistic limit of massive scalar interaction

Consider the scattering  $AB \to AB$  in a theory with two interaction vertices AAC and BBC, both with coupling constant g. There is one Feynman diagram

$$\begin{array}{cccc}
A & g & A \\
p_1 & p_3 & \\
C & q \\
p_2 & p_4 \\
B & g & B \end{array} \qquad \mathcal{M} = \frac{g^2}{(p_1 - p_3)^2 - m_C^2}$$
(4.38)

In order to take the non-relativistic limit we only keep the leading order in the 3-momenta  $p_i = (m, \mathbf{p_i})$ and hence we find for the Lorentz-invariant matrix element

$$\mathcal{M} = \frac{g^2}{-|\mathbf{p}_1 - \mathbf{p}_3|^2 - m_C^2} = 2m_A 2m_B V(|\mathbf{q}|)$$
(4.39)

where we relate it in the last step to the non-relativistic scattering potential with  $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3$  in momentum space

$$V(|\mathbf{q}|) = \frac{-g^2}{4m_A m_B} \frac{1}{|\mathbf{q}|^2 + m_C^2} \,. \tag{4.40}$$

The corresponding potential in real space is

$$V(|\mathbf{r}|) = \int \frac{d^3q}{(2\pi)^3} V(|\mathbf{q}|) e^{i\mathbf{q}\cdot\mathbf{r}}$$
(4.41)

$$= \frac{-g^2}{4m_A m_B} \int \frac{|\mathbf{q}|^2 d|\mathbf{q}| d\Omega}{(2\pi)^3} \frac{e^{i|\mathbf{q}||\mathbf{r}|\cos\theta}}{|\mathbf{q}|^2 + m_C^2}$$
(4.42)

$$= \frac{-g^2}{4m_A m_B (2\pi)^2} \int_0^\infty d|\mathbf{q}| \int_{-1}^1 d\cos\theta \frac{|\mathbf{q}|^2 e^{-i|\mathbf{q}||\mathbf{r}|\cos\theta}}{|\mathbf{q}|^2 + m_C^2}$$
(4.43)

$$= \frac{-g^2}{4m_A m_B (2\pi)^2} \int d|\mathbf{q}| \frac{|\mathbf{q}|^2}{|\mathbf{q}|^2 + m_C^2} \frac{e^{i|\mathbf{q}||\mathbf{r}|} - e^{-i|\mathbf{q}||\mathbf{r}|}}{i|\mathbf{q}||\mathbf{r}|}$$
(4.44)

$$=\frac{-g^2}{i16\pi^2 m_A m_B |\mathbf{r}|} \int_{-\infty}^{\infty} d|\mathbf{q}| \frac{|\mathbf{q}| e^{i|\mathbf{q}||\mathbf{r}|}}{|\mathbf{q}|^2 + m_C^2}$$
(4.45)

$$= \frac{-g^2}{i16\pi^2 m_A m_B |\mathbf{r}|} \int_{-\infty}^{\infty} d|\mathbf{q}| \frac{|\mathbf{q}|e^{i|\mathbf{q}||\mathbf{r}|}}{|\mathbf{q}|^2 + m_C^2}$$
(4.46)

(4.47)

This integral can be solved using the residue theorem. By closing the contour in the upper half plane, we pick up a single pole at  $|\mathbf{q}| = im_C$ . We find the so-called Yukawa potential

$$V(|\mathbf{r}|) = \frac{-g^2}{4m_A m_B} \frac{e^{-m_C|\mathbf{r}|}}{4\pi |\mathbf{r}|}$$
(4.48)

A few comments are in order.

1. The potential has a similar form as the Coulomb potential

$$U_C(r) = \frac{qQ}{4\pi\epsilon_0 r} \tag{4.49}$$

The factors  $g/2m_A$  and  $g/2m_B$  play the role of the charge  $Q/\sqrt{\epsilon_0}$  and the test charge  $q/\sqrt{\epsilon_0}$ .

2. The Coulomb potential describes a static electric force, which is a long-range force, i.e. it does not vanish even for large distances. However the Yukawa potential has an additional factor  $e^{-m_C r}$  which leads to an exponential suppression at large distances. Therefore it describes a short-range force. We can estimate the range of the force by looking at  $m_C r \sim 1$  and thus find that the characteristic range of the force is

$$r \sim \frac{1}{m_C} \tag{4.50}$$

3. In the Standard Model there are also long-range forces like the electromagnetic force and the strong force, which are mediated by their massless force carriers, the photon and gluon. The weak force which is mediated by the massive electroweak W and Z gauge bosons is a short-range force. This can be understood in the same way. The large masses of the W and Z bosons,  $m_W \sim 80$  GeV and  $m_Z \sim 90$  GeV lead to an exponential suppression of the force at large distances. The typical range of the electroweak force is

$$r \sim \frac{1}{m_{W,Z}} \sim \frac{1}{100 GeV} \sim 10^{-18} m$$
 (4.51)

#### 4.6 Standard Model Vertices

In the following we discuss the relevant Standard Model interactions and their vertices schematically. As the matter fields in the Standard Model are spin- $\frac{1}{2}$  Dirac fermions, we have to distinguish between particles and antiparticles and the Feynman rules for the vertices generally contain  $\gamma$ -matrices and are more involved. We denote the direction, in which a particle moves by an arrow. Antiparticles move against the direction of the arrow, e.g.

$$\rightarrow$$
  $\rightarrow$  (4.52)

The diagram on the left-hand side denotes a particle moving from left to right and the diagram on the right-hand side an antiparticle moving from the left to the right. We typically use solid lines for spin- $\frac{1}{2}$  fermions, dashed lines for spin-0 bosons, and curvy lines for spin-1 bosons, like the photon, gluon and electroweak gauge bosons.

Taking this into account, we find for the different gauge interactions in the SM



#### 4.6.1 Common Electromagnetic Processes

• Electron-electron scattering:  $e^-e^- \rightarrow e^-e^-$ 



• Compton scattering  $\gamma e^- \rightarrow \gamma e^-$ 

$$e^{-} \qquad e^{-} \qquad + \qquad e^{-} \qquad e^{-} \qquad (4.59)$$

• Pair annihilation  $e^-e^+ \rightarrow \gamma \gamma$ 

$$e^{-} \qquad e^{-} \qquad e^{-} \qquad e^{-} \qquad e^{-} \qquad \gamma \qquad (4.60)$$

$$e^{+} \qquad e^{+} \qquad e^{+} \qquad \gamma \qquad e^{+} \qquad \gamma$$

The lowest order processes occur at second order in perturbation theory and thus their matrix elements are proportional to

$$\mathcal{M} \propto e^2 = 4\pi\alpha \tag{4.61}$$

where  $\alpha$  denotes the finestructure constant

$$\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137} , \qquad (4.62)$$

which is a dimensionless parameter which quantifies the strength of the electromagnetic interaction. This is consistent with the Coulomb force  $|\mathbf{F}| = kqQ/r^2$  between two particles, which is also proportional to  $e^2$  via the product of the two charges qQ. As  $\sqrt{\alpha} \ll 1$  perturbation theory works well.

#### 4.6.2 Common Weak Processes

For weak interactions we distinguish between charged current interactions and neutral current interactions. Charged current processes are mediated by the W boson. Relevant charged current processes include:

• Muon decay:  $\mu^- \to e^- \nu_\mu \bar{\nu}_e$ 

• Pion decay:  $\pi^+(u\bar{d}) \rightarrow \mu^+ + \nu_\mu$ 

$$\begin{array}{c} u \\ \overline{d} \end{array} \begin{array}{c} W^+ \\ g_W \\ g_W \\ \nu_\mu \end{array}$$
(4.64)

• neutron  $\beta$  decay:  $n(udd) \rightarrow p(uud) + e^- + \bar{\nu}_e$ . Two of the quarks do not participate in the process. They are so-called spectator quarks. The underlying quark level process is  $d \rightarrow u + e^- + \bar{\nu}_e$ 



Neutrino electron scattering is one common neutral current interaction:  $\nu_{\mu}e^{-} \rightarrow \nu_{\mu}e^{-}$ 



The similar process  $\nu_e e^- \rightarrow \nu_e e^-$  is mediated by both Z and W bosons, i.e. there are both a charged and neutral current contribution.



Similarly to electromagnetic interactions we can estimate the strength of the weak processes. We will focus on the charged current process. Recall from Sec. 4.2, the scattering amplitude of a  $2 \rightarrow 2$  scattering process which is mediated by a massive virtual particle can be written as

$$\mathcal{M} \sim \frac{g_W^2}{q^2 - m_W^2} \tag{4.68}$$

where  $q^2 = q_{\mu}q^{\mu}$  is the 4-momentum transfer. Weak decay processes have typically small 4-momentum transfer,  $q^2 \ll m_W^2$ , e.g. in muon decay the momentum transfer is  $\leq m_{\mu}^2$ . For those processes it is justified to take the limit  $q^2 \ll m_W^2$ . In this limit we obtain for the matrix element

$$\mathcal{M} \stackrel{q^2 \ll m_W^2}{\Longrightarrow} \frac{-g_W^2}{m_W^2} \tag{4.69}$$

and they are commonly parameterized in terms of the Fermi constant

$$G_F \equiv \frac{\sqrt{2}}{8} \frac{g_W^2}{m_W^2} \,. \tag{4.70}$$

The Fermi constant is measured in muon decay to be  $G_F = 1.166 \times 10^{-5} \text{GeV}^{-2}$  and thus  $g_W \simeq 0.65$ . The corresponding fine structure constant is

$$\alpha_W \equiv \frac{g_W^2}{4\pi} \simeq \frac{1}{30} \tag{4.71}$$

and thus it is larger than the electromagnetic fine structure constant. Note, weak interactions are not weak, because the weak fine structure constant is small, but because of the heavy mass of the electroweak gauge bosons, which imply that scattering amplitudes are suppressed by  $m_W^{-2}$ . This results in a short-range interaction.

#### 4.6.3 Strong interactions

As the fine structure constant for strong interactions is large,  $\alpha_s \simeq 1$  at low energies, perturbation theory breaks down.

Note however, that the fine structure constants are not constants in a quantum field theory, but depend on the energy scale. This is illustrated in Fig. 8. Fig. 8a illustrates how the energy dependence of the fine structure constants emerges from the polarizability of the quantum field theory vacuum. In a quantum field theory the vacuum is not empty, but it is always possible to create particle antiparticle pairs, which annihilate again. This behaves like a polarizable medium and leads to the screening of charges of charged particles. For quantum electrodynamics, the effective charge gets smaller and smaller the smaller the energy (and larger the distance which is probed). The effect is reversed for strong interactions, which become stronger at small distances, due to self-interactions of gluons, the force carriers of strong interactions. The energy dependence of the different fine structure constants is shown in Fig. 8b in the Standard Model (left) and the minimal supersymmetric Standard Model (MSSM) shown in the plot on the right-hand side. Note that the values of the fine structure constants are almost equal at a high energy scale of  $O(10^{15})$  GeV and they actually do in the MSSM, which implies that the 3 different forces in the SM have the same strength at high energy and suggests that the forces could be unified and be described by a Grand Unified Theory (GUT).

#### 4.7 Example: quantum electrodynamics (QED)

So far we did an explicit calculation in the ABC theory and we surveyed the relevant interactions in the Standard Model. This section will discuss one more example in quantum electrodynamics. We first summarise the relevant Feynman rules for  $i\mathcal{M}^8$ :

- 1. Dirac fermions by solid lines and gauge bosons by wavy lines. Charged particles have an arrow on the line, which indicates the direction of the flow of charge.
- 2. Ingoing (outgoing) external gauge bosons by the polarisation vector of the gauge boson  $\epsilon_{\mu}$  ( $\epsilon_{\mu}^{*}$ ), ingoing (outgoing) fermions by u ( $\bar{u} = u^{\dagger}\gamma^{0}$ ) spinor and ingoing (outgoing) antifermions by the

<sup>&</sup>lt;sup>8</sup>Note that in contrast to before, we calculate  $i\mathcal{M}$  instead of  $\mathcal{M}$ .



(a) Polarizability of the quantum field theory vacuum leads to running of the couplings. Taken from C. Steinwachs, "Effective Action and Renormalization Group".

(b) Energy dependence of the inverse fine structure constants in the Standard Model (left) and the Minimal Supersymmetric Standard Model (right).

Figure 8: Quantum field theory vacuum and the energy dependence of couplings.

corresponding  $\bar{v}(v)$  spinor.

- 3. At every vertex 4-momentum is conserved.
- 4. An internal line for a particle with 4-momentum  $p^{\mu}$  (pointing from left to right) corresponds to a propagator

$$\mu \sim \nu \qquad \frac{-i\eta^{\mu\nu}}{p^2 - m^2 + i\epsilon} \qquad \qquad \longrightarrow \qquad \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon} \tag{4.73}$$

Note that virtual particles do not generally satisfy the dispersion relation  $p_{\mu}p^{\mu} = m^2$ . The  $+i\epsilon$  ensures the causality.  $\epsilon$  is taken to zero at the end of the calculation.

5. In quantum electrodynamics there is only one relevant interaction<sup>9</sup>

$$-ie\gamma^{\mu} \qquad (4.74)$$

6. Integrate over any free internal 4-momenta  $k_i$ 

$$\int \frac{d^4k_i}{(2\pi)^4} \tag{4.75}$$

7. If the Feynman diagram exhibits any symmetry, divide by the number of possible symmetric configurations (symmetry factor).

<sup>&</sup>lt;sup>9</sup>Later in the course, we will see that the corresponding term in the Lagrangian is  $\mathcal{L} = -eA_{\mu}\bar{\psi}\gamma^{\mu}\psi$ .

- 8. As fermions are described by spinors and to properly take into account the matrix structure of the  $\gamma$ -matrices, fermion lines are read in the opposite direction of the arrows.
- 9. Sum over the expressions of all possible different diagrams. If external fermion lines are exchanged, there is a relative minus sign between diagrams for every exchange of fermions.
- 10. Closed loops of fermions give an additional factor -1.

As an example we will calculate the scattering amplitude for  $e^-e^+ \rightarrow e^-e^+$ . The leading order process occurs at order  $e^2$  like the other electromagnetic processes which we discussed. Using the interaction vertex and the propagators, we can construct two diagrams at leading order



Using the Feynman rules for quantum electrodynamics, we can evaluate the first diagram

$$e^{-(1)} e^{-(3)} i\mathcal{M}_{1} = [\bar{u}_{3}ie\gamma^{\mu}v_{4}][\bar{v}_{2}ie\gamma^{\nu}u_{1}]\frac{-ig_{\mu\nu}}{(p_{1}+p_{2})^{2}+i\epsilon}, \qquad (4.77)$$

$$e^{+}(2) e^{+}(4)$$

where the subscripts on the spinors u, v (momenta  $p_i$ ) denote which of the external fermions it belongs to. Keep in mind that the order of the spinors and  $\gamma^{\mu}$  matrices matters:  $\gamma^{\mu}$  are matrices in spinor space,  $u_i$  and  $v_i$  are column vectors in spinor space and  $\bar{u}_i$  and  $\bar{v}_i$  are row vectors in spinor space. It is convenient to introduce Mandelstam variables

$$s = (p_1 + p_2)^2$$
  $t = (p_1 - p_3)^2$   $u = (p_1 - p_4)^2$  (4.78)

and rewrite the matrix element as

$$i\mathcal{M}_1 = [\bar{u}_3 i e \gamma^\mu v_4] [\bar{v}_2 i e \gamma^\nu u_1] \frac{-ig_{\mu\nu}}{s+i\epsilon} , \qquad (4.79)$$

where  $\gamma_{\mu} \equiv g_{\mu\nu} \gamma^{\nu}$ . Similarly we find for the second diagram

$$e^{-(1)} \xrightarrow{} e^{-(3)} e^{+(2)} \xrightarrow{} e^{+(4)} i\mathcal{M}_{2} = -[\bar{u}_{3}ie\gamma^{\mu}u_{1}][\bar{v}_{2}ie\gamma^{\nu}v_{4}] \frac{-ig_{\mu\nu}}{(p_{1}-p_{3})^{2}+i\epsilon} = ie^{2}\frac{[\bar{u}_{3}\gamma^{\mu}u_{1}][\bar{v}_{2}\gamma_{\mu}v_{4}]}{t+i\epsilon}$$

$$(4.80)$$

The total matrix element is then given by

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = e^2 \left( \frac{[\bar{u}_3 \gamma^\mu v_4] [\bar{v}_2 \gamma_\mu u_1]}{s + i\epsilon} - \frac{[\bar{u}_3 \gamma^\mu u_1] [\bar{v}_2 \gamma_\mu v_4]}{t + i\epsilon} \right) . \tag{4.81}$$

Obviously, these are only the lowest-order diagrams which contribute to the scattering  $e^-e^+ \rightarrow e^-e^+$ . We could for example exchange multiple photons or the photon could split into a virtual  $e^+e^-$  pair which then recombines into a photon.

## 5 Classical field theory

The Klein-Gordon and Dirac equations can also be derived from the stationary action principle. For definiteness we will consider the Klein-Gordon equation in the following which can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 \tag{5.1}$$

and thus the action  $S = \int dt \int d^3x \mathcal{L}$ . Consider a variation of the action with respect to the field  $\phi$ and the coordinates  $x^{\mu}$ 

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} \qquad \phi'(x) = \phi(x) + \delta \phi(x) .$$
 (5.2)

In the absence of an explicit x-dependence, the variation of the action yields

$$\delta S = \int_{R} d^{4}x \left[ \mathcal{L}(\phi', \partial_{\mu}\phi') - \mathcal{L}(\phi, \partial_{\mu}\phi) \right]$$
(5.3)

$$= \int_{R} d^{4}x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta (\partial_{\mu} \phi) \right)$$
(5.4)

$$= \int_{R} d^{4}x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right) \delta\phi + \int_{R} d^{4}x \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta\phi$$
(5.5)

$$= \int_{R} d^{4}x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)}\right) \delta \phi + \int_{\partial R} d\sigma_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi$$
(5.6)

Using Gauss theorem the second term can be rewritten as a surface integral. If we demand that there is no variation,  $\delta x^{\mu} = 0$  and  $\delta \phi = 0$ , on the boundary  $\partial R$ , the second term vanishes and we obtain the Euler-Lagrange equations

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)}$$
(5.7)

For a real scalar field the Euler-Lagrange equation is

$$0 = \frac{\delta \mathcal{L}}{\delta \phi} - \partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} \phi} = -\partial_{\mu} \partial^{\mu} \phi - m^2 \phi$$
(5.8)

which is exactly the Klein-Gordon equation. The conjugate momentum to the field  $\phi$  is

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \tag{5.9}$$

and the Hamiltonian density is obtained by a Legendre transformation

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} . \tag{5.10}$$

#### 5.1 Noether's theorem

Similarly to the classical mechanics of point particles, Noether's theorem establishes a connection between continuous symmetries and conserved quantities. We will only consider transformations of the fields  $\phi_i$  and not spacetime transformations. One example of such a transformation is  $\phi \to e^{-i\epsilon}\phi$ or more generally for i = 1, ..., N fields

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta \phi_i(x) = \phi_i(x) - i\epsilon_a F^a_i[\phi_j(x)] .$$
(5.11)

 $\epsilon_a$  are real (x-independent) parameters,  $F_i^a = \frac{\partial \delta \phi_i}{\partial \epsilon_a}|_{\epsilon_a=0}$  are functions of the fields and we neglected terms of order  $\epsilon^2$  and higher. From Eq. (5.6) we find that the symmetry of the action, implies that the second term vanishes for an arbitrary surface  $\partial R$  and thus the integrand has to vanish itself

$$\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi = \epsilon_a \partial_{\mu} J^{\mu} , \qquad (5.12)$$

where we defined the 4-current density

$$J^{\mu,a} = -i\frac{\partial \mathcal{L}}{\partial \partial_{\mu}\phi_{i}}F_{i}^{a}.$$
(5.13)

The divergence of the 4-current density is

$$\partial_{\mu}J^{\mu,a} = -i\left(\partial_{\mu}\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_{i}}\right)F_{i}^{a} - i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_{i}}\partial_{\mu}F_{i}^{a} = -i\left(\frac{\partial\mathcal{L}}{\partial\phi_{i}}\right)F_{i}^{a} - i\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi_{i}}\partial_{\mu}F_{i}^{a} , \qquad (5.14)$$

where we used the product rule in the first step and the equations of motion in the second step. Thus we find a version of Noether's theorem: For the global (spacetime-independent) symmetry  $\phi_i \rightarrow \phi'_i = \phi_i + \delta \phi_i$ , the 4-current  $J^{\mu,a}$  is conserved,  $\partial_{\mu} J^{\mu,a} = 0$ . For a conserved current the charge

$$Q^{a}(t) = \int d^{3}x J_{0}^{a}(x)$$
(5.15)

is time independent, i.e. a constant of motion.

## 6 Standard Model Lagrangian and its symmetries

We already encountered symmetries of the Lagrangian, like a U(1) symmetry of the complex scalar field. Symmetries are a very important ingredient in particle physics and characterize all interactions. Equally important is symmetry breaking, because it explains the masses of all matter (fermions) and the electroweak gauge bosons (W and Z boson). In the following we will discuss it at an elementary (classical) level to get an understanding of the symmetries in particle physics.

#### 6.1 Dirac Lagrangian

The Dirac equation is the equation of motion of a field theory. The Lagrangian corresponding to the Dirac equation is given by

$$\mathcal{L} = \bar{\psi}(i\partial \!\!\!/ - m)\psi , \qquad (6.1)$$

where  $\psi_i$  and  $\bar{\psi}_i$  can be considered as independent fields (Complex fields have two independent real components.). The equation of motion for  $\bar{\psi}$  leads to the Dirac equation

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} = \left[ (i \partial - m) \psi \right]_i \tag{6.2}$$

Exercise:

- Show that the Lagrangian (6.1) is invariant under a global (x independent) symmetry  $\psi \to e^{-i\alpha}\psi$ .
- Show that the associated conserved current is

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi . \tag{6.3}$$

• The conserved charge is

$$Q = \int d^3x \psi^{\dagger} \psi \;. \tag{6.4}$$

This motivates the ad-hoc definition of the continuity equation in Sec. 3.2.

#### 6.2 Gauge symmetries

What do we have to change in order to obtain a Lagrangian which is invariant under local (xdependent) transformations,  $\psi \to \psi' = e^{-i\alpha(x)}\psi$ ? An infinitesimal transformation is  $\delta\psi = -i\alpha(x)\psi$ and  $\delta\bar{\psi} = +i\alpha(x)\bar{\psi}$  and thus a local transformation leads to the following variation of the Lagrangian

$$\delta \mathcal{L} = \delta \bar{\psi} (i \partial \!\!\!/ - m) \psi + \bar{\psi} (i \partial \!\!\!/ - m) \delta \psi$$
(6.5)

$$= i\alpha(x)\bar{\psi}(i\partial - m)\psi + \bar{\psi}(i\partial - m)(-i\alpha(x)\psi$$
(6.6)

$$= i\alpha(x)\bar{\psi}(i\partial \!\!\!/ - m)\psi + -i\alpha(x)\bar{\psi}(i\partial \!\!\!/ - m)\psi + \bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\alpha(x)$$
(6.7)

$$=j^{\mu}\partial_{\mu}\alpha , \qquad (6.8)$$

where we used product rule in the second line and third line and cancelled terms in the last line. The kinetic term (the term with the derivative) led to the non-trivial variation of the Lagrangian. We can make the Lagrangian invariant under a local (or gauge) transformation by replacing the partial derivative by a *covariant derivative*  $D_{\mu}$  which is defined as  $D_{\mu} \equiv \partial_{\mu} - ieA_{\mu}$  where  $A_{\mu}$  is a field which transforms as a Lorentz 4-vector. The covariant derivative should satisfy

$$D_{\mu}\psi \to e^{-i\alpha}D_{\mu}\psi$$
 . (6.9)

How does  $A_{\mu}$  transform if  $\psi \to e^{-i\alpha}\psi$ ? Consider a general transformation  $A_{\mu} \to A'_{\mu}$ 

$$D_{\mu}\psi \to (\partial_{\mu} - ieA'_{\mu})e^{-i\alpha}\psi = e^{-i\alpha\psi}(\partial_{\mu} - ieA'_{\mu} - i\partial_{\mu}\alpha)\psi$$
(6.10)

Hence  $A_{\mu}$  transforms as follows

$$A_{\mu} \to A'_{\mu} = A_{\mu} - \frac{1}{e} \partial_{\mu} \alpha .$$
(6.11)

which is exactly the gauge transformation in electrodynamics. The vector field  $A^{\mu}$  is a gauge field. It could for example describe the electromagnetic field. In electromagnetism the time component corresponds to the scalar potential  $A^0 = \phi$  and the spatial components form the vector potential  $(A^i) = \mathbf{A}^{10}$ 

Note, the form of the covariant derivative explains why antiparticles have opposite charge. Consider the Dirac equation with a covariant derivative

$$(i\not\!\!D - m)\psi = 0. ag{6.12}$$

Particle spinors of the form  $\psi = ue^{-ipx}$  are solutions of the Dirac equation, if

$$(p + eA - m)u = 0. (6.13)$$

The corresponding antiparticle spinors  $v(p^{\mu}) = u(-p^{\mu})$  are obtained by replacing  $p^{\mu} \to -p^{\mu}$  and thus satisfy

$$(p - eA + m)v = 0.$$
 (6.14)

Comparing the two algebraic equations for u and v, we see that the coupling of the spinor to the electromagnetic field changed sign and thus for a particle with charge q the corresponding antiparticle has charge -q.

#### 6.3 Lagrangian of electrodynamics

From the gauge field  $A^{\mu}$  we can define the *field strength tensor* 

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} , \qquad (6.15)$$

<sup>&</sup>lt;sup>10</sup>It is a contravariant vector field, i.e.  $A^{\mu} = (\phi, \mathbf{A})$  and  $A_{\mu} = (\phi, -\mathbf{A})$ .

which is invariant under gauge transformations

$$F^{\mu\nu} \to \partial^{\mu} A^{\prime\nu} - \partial^{\nu} A^{\prime\mu} \tag{6.16}$$

$$=\partial^{\mu}\left(A^{\nu}-\frac{1}{e}\partial^{\nu}\alpha\right)-\partial^{\nu}\left(A^{\mu}-\frac{1}{e}\partial^{\mu}\alpha\right)$$
(6.17)

$$=\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = F^{\mu\nu} , \qquad (6.18)$$

where we used that the  $\alpha(x)$  is a smooth function such that we can permute the derivatives. The field strength tensor is antisymmetric in its indices,  $F^{\mu\nu} = -F^{\nu\mu}$ , and thus the diagonal entries vanish  $F^{\mu\mu} = 0$ . We find for the mixed time and spatial components

$$(F^{i0}) = \partial^i A^0 - \partial^0 A^i = -\nabla \phi - \partial_t \mathbf{A} , \qquad (6.19)$$

where we expressed  $A^{\mu}$  in terms of the scalar potential and the vector potential **A** in the last line. A comparison with electrodynamics shows that  $F^{i0}$  is exactly the electric field  $\mathbf{E} = (F^{i0})$ . Moving on to the spatial components of the field strength tensor we find

$$F^{ij} = \partial^i A^j - \partial^j A^i \tag{6.20}$$

$$= (\delta_m^i \delta_n^j - \delta_n^i \delta_m^j) \partial^m A^n \tag{6.21}$$

$$=\epsilon^{kij}\epsilon_{kmn}\partial^m A^n \tag{6.22}$$

$$= -\epsilon^{ijk} B^k , \qquad (6.23)$$

where we used introduced the totally antisymmetric (Levi-Civita) tensor  $\epsilon^{ijk} = \epsilon_{ijk}$  in the third line using the relation  $\epsilon^{imn}\epsilon_{ijk} = \delta^m_j \delta^n_k - \delta^m_k \delta^n_j$ . In the last line we used  $(\partial^k) = -\nabla$  and the relation between the magnetic field **B** and the vector potential **A** 

$$\mathbf{B} = \operatorname{curl} \mathbf{A} = \nabla \wedge \mathbf{A} = \left(\epsilon^{ijk} \frac{\partial A^k}{\partial x^j}\right) \,. \tag{6.24}$$

Concluding we can express the field strength tensor in terms of the electric  $\mathbf{E}$  and magnetic fields  $\mathbf{B}$ 

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad \text{and} \quad (F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$
(6.25)

In order to construct a Lagrangian, we have to find gauge and Lorentz-invariant combinations of the fields. One expression is obtained by contracting the field strength tensor with itself<sup>11</sup>

$$F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2F_{0i}F^{0i} + F_{ij}F^{ij} = -2\mathbf{E}^2 + \epsilon_{ijk}\epsilon^{ijl}B^kB^l = -2\mathbf{E}^2 + 2\mathbf{B}^2 . \quad (6.26)$$

<sup>&</sup>lt;sup>11</sup>There is in principle a second term, constructed out of the field strength tensor and its dual  $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ . However  $\tilde{F}_{\mu\nu}F^{\mu\nu}$  is a total derivative and a topological term, which does not contribute to the equations of motion.

The Lagrangian of electrodynamics is thus given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not\!\!D - m)\psi$$
(6.27)

with  $D_{\mu} = \partial_{\mu} - ieA_{\mu}$ . The Euler Lagrange equations are

$$\partial_{\mu}F^{\mu\nu} = -e\bar{\psi}\gamma^{\mu}\psi = -ej^{\mu} \qquad (i\not\!\!\!D - m)\psi = 0 , \qquad (6.28)$$

where we used

$$\frac{\partial \mathcal{L}}{\partial(\partial_{\rho}A_{\sigma})} = -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_{\rho}A_{\sigma})} = -F^{\rho\sigma}.$$
(6.29)

Note that the Lagrangian for electrodynamics does not contain a term of the form  $A_{\mu}A^{\mu}$ , which would be a mass term for the gauge field similar to the mass term of a scalar field. This term is forbidden by the U(1) symmetry. As  $\delta A_{\mu} = -\frac{1}{e}\partial_{\mu}\alpha$ , the quadratic term transforms as

$$\delta(A_{\mu}A^{\mu}) = A_{\mu}\delta A^{\mu} + \delta A_{\mu}A^{\mu} = -\frac{1}{e}A_{\mu}\partial^{\mu}\alpha - \frac{1}{e}\partial_{\mu}\alpha A^{\mu} = -\frac{2}{e}A_{\mu}\partial^{\mu}\alpha \neq 0.$$
(6.30)

From this we conclude that the gauge boson is massless due to the gauge symmetry.

#### 6.4 The Standard Model Lagrangian

The construction of the Lagrangian for the other gauge symmetries of the Standard Model follows the same steps, but it is more involved due to the non-Abelian nature of the groups SU(2) and SU(3) and we will not discuss it here in detail, but only sketch a points. For non-Abelian gauge theories, the vector fields become matrix valued, e.g. for SU(2), they are represented by hermitian traceless  $2 \times 2$  matrices  $A_{\mu} = A_{\mu}^{a} \frac{\sigma^{a}}{2}$ , similarly the field strength tensor is matrix valued. In general, the field strength tensor can be constructed out of the covariant derivatives  $D_{\mu} = \partial_{\mu} + igA_{\mu} = \partial_{\mu} + igA_{\mu}^{a}T^{a}$  as

$$igF_{\mu\nu} = [D_{\mu}, D_{\nu}] = ig(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu}, A_{\nu}])$$
(6.31)

and contains an additional term  $[A_{\mu}, A_{\nu}]$  due to the non-Abelian nature of the symmetry group.

We are now in the position to understand the basic properties of the Standard Model Lagrangian. The relevant parts of the Standard Model Lagrangian are given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L i \mathcal{D}\psi_L + \bar{\psi}_R i \mathcal{D}\psi_R + |D_\mu\phi|^2 - V(\phi) - \left(y\bar{\psi}_L\psi_R\phi + y^*\bar{\psi}_R\psi_L\phi^*\right) , \qquad (6.32)$$

where  $F_{\mu\nu}$  stands for the field strength tensors of the different gauge fields, D denotes the covariant derivatives and  $\psi_L \equiv P_L \psi$  ( $\psi_R \equiv P_R \psi$ ) denotes a left-chiral (right-chiral) fermion,  $\phi$  describes the Higgs field, a spin-0 scalar field.

## 7 Masses in the Standard Model and the Higgs mechanism

In the Standard Model, three of the force carriers are massive and thus mediate a short-range force. In this section, we will discuss how they (and all Standard Model fermions) obtain their mass. We however will work with a simpler theory and only consider how U(1) symmetry is spontaneously broken instead of electroweak symmetry  $SU(2) \times U(1)$ .

#### 7.1 Spontaneous symmetry breaking of a global symmetry

Consider a real scalar field  $\phi$  with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) .$$
(7.1)

We consider the following potential for the field

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^2 + \frac{\lambda}{4}\phi^4 , \qquad (7.2)$$

where  $\lambda$  and  $\mu^2$  are real constants. The Lagrangian is invariant under a reflection symmetry  $\phi \to -\phi$ , i.e. a  $Z_2$  symmetry. The corresponding Hamiltonian is



Figure 9: Scalar field potential

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$
(7.3)

We are interested in the lowest energy field configuration  $\phi(x)$ . As the derivative terms  $\dot{\phi}^2$  and  $\nabla \phi)^2$  are squared, they have to vanish separately, which implies that the field should be constant,  $\phi(x) = \phi_0$ . The value  $\phi_0$  is determined by the potential. Let us assume that  $\mu^2 > 0$  and thus the minimum of the potential is not at a vanishing field value. In this case the quadratic term  $(\mu^2)$  has the wrong sign. In Fig. 9 we plot the potential vs the field value. Note that the minimum of the potential is not at  $\phi_0 = 0$ , but at  $\phi_0^2 = \mu^2 / \lambda$ . There are two degenerate minima. In a quantum field theory tunneling between the different minima (vacua in quantum field theory jargon) is exponentially suppressed and thus the physical system will generally choose one vacuum and thus the vacuum does not respect the  $Z_2$  symmetry of the Lagrangian. Without loss of generality we choose  $\phi_0 = \mu / \sqrt{\lambda}$ .

It is convenient to expand the field around the vacuum and define the field  $\eta(x)$  in terms of  $\phi(x) = \phi_0 + \eta(x)$ . The potential as a function of the field  $\eta$  is given by

$$V(\eta) = \mu^2 \eta^2 \mp \mu \sqrt{\lambda} \eta^3 + \frac{\lambda}{4} \eta^4 - \frac{\mu^4}{4\lambda} .$$
(7.4)

A comparison with a free real scalar field shows that the mass of the  $\eta$  is  $m_{\eta}^2 = 2\mu^2$ . Clearly, the potential of  $\eta$  is not symmetric under  $\eta \to -\eta$ . This is called *spontaneous symmetry breaking*. A similar discussion applies to a complex scalar field with a continuous global U(1) symmetry.

#### 7.2 Abelian Higgs mechanism

For a gauge symmetry the discussion changes. Consider a complex scalar field with a U(1) gauge symmetry and covariant derivative  $D_{\mu} = \partial_{\mu} + igA_{\mu}$ . The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_{\mu}\phi)^{\dagger}D^{\mu}\phi - V(\phi^{\dagger}\phi) .$$
(7.5)

We do specify the potential in the following, but assume that the minimum is not at  $\phi(x) = 0$ , but shifted to a constant value  $\phi(x) = \phi_0$ . If the vacuum does not respect the U(1) symmetry, the symmetry will be broken as before and we have to shift the field to expand around the new vacuum

$$\phi(x) = \frac{\phi_0 + \eta(x)}{\sqrt{2}} e^{i\alpha(x)} .$$
(7.6)

The factor  $1/\sqrt{2}$  is introduced, such that the scalar kinetic terms are normalized like the real scalar field with  $\frac{1}{2}\partial_{\mu}\eta\partial^{\mu}\eta$ . The potential is given by  $V((\phi_0 + \eta)^2/2)$  and thus does not depend on the phase factor  $\alpha$ . In order to evaluate the kinetic term, we first evaluate the covariant derivative

$$D_{\mu}\phi = \left(\partial_{\mu} + igA_{\mu}\right)\left(\frac{\phi_{0} + \eta}{\sqrt{2}}e^{i\alpha}\right) = \frac{1}{\sqrt{2}}\left(e^{i\alpha}\partial_{\mu}\eta + (\phi_{0} + \eta)e^{i\alpha}i\partial_{\mu}\alpha\right) + igA_{\mu}\left(\frac{\phi_{0} + \eta}{\sqrt{2}}e^{i\alpha}\right)$$
(7.7)

$$=e^{i\alpha}\left[\left(\partial_{\mu}+i(gA_{\mu}+\partial_{\mu}\alpha)\right)\frac{\eta}{\sqrt{2}}+i(gA_{\mu}+\partial_{\mu}\alpha)\frac{\phi_{0}}{\sqrt{2}}\right].$$
(7.8)

We use a gauge transformation  $A_{\mu} \to A_{\mu} - \frac{1}{g} \partial_{\mu} \alpha$  to absorb  $\partial_{\mu} \alpha$  in the gauge field and thus obtain after the gauge transformation

$$D_{\mu}\phi = \left[ \left(\partial_{\mu} + igA_{\mu}\right)\frac{\eta}{\sqrt{2}} + igA_{\mu}\frac{\phi_{0}}{\sqrt{2}} \right]$$
(7.9)

The factor of  $e^{i\alpha}$  is simultaneously removed by the gauge transformation, since  $\phi \to e^{-i\alpha}\phi$ . Thus up to quadratic order in the fields we obtain

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_{\mu}\eta\partial^{\mu}\eta + \frac{g^{2}\phi_{0}^{2}}{2}A_{\mu}A^{\mu} - \frac{1}{2}m_{\eta}^{2}\eta^{2} + \dots$$
(7.10)

Note in particular the second last term which is quadratic in the gauge field  $A_{\mu}A^{\mu}$ . It corresponds to a mass term of the gauge field  $m_A^2 = g^2 \phi_0^2$ . This is one example of the *Higgs mechanism*, where spontaneous symmetry breaking leads to massive gauge boson.

If the scalar field interacts with fermions via

$$\mathcal{L}_{\text{Yukawa}} = -y\phi\bar{\psi}_L\psi_R - y\phi^{\dagger}\bar{\psi}_R\psi_L = -y\frac{\phi_0}{\sqrt{2}}(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) + \dots = -y\frac{\phi_0}{\sqrt{2}}\bar{\psi}\psi + \dots , \qquad (7.11)$$

where y is a real number, it will generate a Dirac mass term  $m_{\psi} = y\phi_0/\sqrt{2}$  for the Dirac fermion  $\psi$ . The Higgs mechanism is realized in the Standard Model of particle physics. It generates the masses of the gauge bosons and elementary charged fermions.

#### 7.3 Standard Model

Symmetries are a crucial ingredient in particle physics. The gauge symmetry group of the Standard Model is  $SU(3)_c \times SU(2)_L \times U(1)_Y$ , where  $SU(3)_c$  desribes strong interactions and  $SU(2)_L \times U(1)_Y$  describes electroweak interactions. The symmetry properties of all particles of the SM are given in Tab. 1. The number in the columns of  $SU(3)_c$  and  $SU(2)_L$  indicates the dimension of the representation, e.g. 1 means that the field does not transform under the symmetry, while a 2 or 3 indicates that the field transforms like a 2 or 3 dimensional representation. The 2 dimensional representation of  $SU(2)_L$  is typically called isospin in analogy to spin, because it is exactly the same mathematical structure as spin and there are two components, the isospin  $\frac{1}{2}$  component of  $Q_L$  is the left-handed up-type quark and the isospin  $-\frac{1}{2}$  component of  $Q_L$  is the left-handed down-type quark or  $Q_L = (u_L, d_L)^T$ , while  $L_L = (\nu_L, e_L)^T$ . The last column indicates the hypercharge Y of each field, e.g. a field  $\phi$  with hypercharge Y transforms as  $\phi \to e^{iY\alpha}\phi$  under  $U(1)_Y$ . The SM gauge symmetry is spontaneously

	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$Q_L$	3	2	$\frac{1}{6}$
$u_R$	3	1	$\frac{2}{3}$
$d_R$	3	1	$-\frac{1}{3}$
$L_L$	1	2	$-\frac{1}{2}$
$e_R$	1	1	-1
Higgs H	1	2	$\frac{1}{2}$

Table 1: Fermions and scalars of the SM.

broken to  $SU(3)_c \times U(1)_{em}$ , more precisely electroweak symmetry  $SU(2)_L \times U(1)_Y$  is spontaneously broken to electromagnetism  $U(1)_{em}$ . Via the Higgs mechanism the gauge bosons of the broken part of the symmetry  $(Z, W^{\pm}$  gauge bosons) obtain their mass from the Higgs mechanism. Also all charged fermions obtain their mass from their interaction with the Higgs boson. The electric charge Q is related to isospin  $T_3$  and hypercharge Y by  $Q = T_3 + Y$ .



Figure 10:  $b \rightarrow s\gamma$ 

Note that left-handed fermions and right-handed fermions transform differently under the gauge group, i.e. the theory is not symmetric under parity. Hence it is not possible to write down a Dirac mass term  $\bar{Q}_L u_R$  and the Higgs boson is required to write down a gauge-invariant term in the Lagrangian. The couplings of the SM fermions to the Higgs bosons are schematically given by

$$-\mathcal{L}_{\text{Yukawa}} = y_u Q_L H^* u_R + y_d Q_L H d_R + y_e L_L H e_R + h.c.$$
(7.12)

As the couplings of the Higgs field to fermion fields generate the fermion masses, the coupling of the Higgs particle to fermions is proportional to the fermion mass, and there are no couplings of the Higgs bosons which couple different fermions, e.g. there are Higgs decays  $H \to \bar{b}b$ , but not  $H \to \bar{b}s$ . Similarly, the Z boson always couples to a pair of fermions,  $Z \to e^+e^-$ , but not  $Z \to e^+\mu^-$ , which would require a coupling of the Z boson to two different fermions. In the Standard Model there are no flavour-changing neutral currents [FCNC] (interactions) at tree-level. At higher order, it is possible to have FCNC, because the W boson (charged current) does not conserve flavour.

The W boson couples up-type quarks with down-type quarks  $W^- \to \bar{u}d$  and more generally  $W^- \to \bar{u}_i d_j$  and charged leptons to neutrinos  $W^- \to \bar{\nu}_e e^-$ . While the W boson mediates interactions between different generations of quarks in the SM, like  $b \to s\gamma$  (see Fig. 10), it does not mediate interactions between different generations of leptons. In the SM there are no interactions which convert a quark into a lepton, more precisely, the total number of quarks minus the total number of antiquarks is conserved, similarly for leptons. In fact there are accidental symmetries of the SM Lagrangian, which have not been imposed, but appear as a result of the demanded gauge symmetries of the SM. In particular, the lepton flavours are individually conserved, i.e.

$$(L_{La}, e_{Ra}) \to e^{i\alpha_a}(L_{La}, e_{Ra}) , \qquad (7.13)$$

where a labels the lepton flavours  $(e, \mu, \tau)$ , are symmetries of the Lagranigan and also baryon number is conserved, because

$$(Q_L, u_R, d_R) \to e^{i\alpha}(Q_L, u_R, d_R) \tag{7.14}$$

is a symmetry of the Lagrangian. Note that in this case it does not hold for individual flavours, i.e. there is only one symmetry.

The SM symmetries together with the SM particle content clearly specify all possible interactions. Similar to 4-momentum conservation at each vertex, which is a result of translation invariance, at each vertex all (unbroken) charges are conserved. In the SM these charges are the electric charge and



(c) Weak interactions. Only left-handed particles couple to  $W^{\pm}$ . Electric charge is conserved at each charged current vertex. In charged current interactions the quarks can be from different generations. In neutral current interactions, both fermions are from the same generation.



(d) Higgs interactions.  $f=u,d,\ell.~\langle H\rangle$  denotes the Higgs vacuum expectation value.



(e) Selection of gauge boson self interactions. Note that the photon does not have self interactions.

Figure 11: Relevant SM interaction vertices

color charge, which we did not discuss, because it requires more group theory. Furthermore, also all accidental charges are conserved, individual lepton number and the baryon number is conserved at each vertex. Finally, some parts of a theory may respect further conditions, e.g. in the SM there are no FCNCs as discussed above and thus both the Higgs and Z boson both always couple to a pair of the same fermion and never couple to two different fermions. Taking all of this into account, we list a selection of all relevant SM interactions in Fig. 11.

## 8 A taste of quantum field theory

As we previously discussed the Klein-Gordon equation is solved by plane waves. Thus the general solution is given by their superposition. In order to avoid difficulties with infinities, we will consider the Klein-Gordon equation in a box, a finite volume V, and impose periodic boundary conditions  $\phi(0, y, z, t) = \phi(L, y, z, t)$ , etc., where L is the size of the box in x-direction. The wave vectors are of the form  $\mathbf{k} = \frac{2\pi}{L} \mathbf{n}$  with  $n_i = 0, \pm 1, \ldots$ . In this case on

$$\phi(x^{\mu}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}} \left( a(\mathbf{k})e^{-ik_{\mu}x^{\mu}} + a^*(\mathbf{k})e^{+ik_{\mu}x^{\mu}} \right)$$
(8.1)

The coefficients of the positive energy solution  $a(\mathbf{k})$  and the negative energy solution  $a^*(\mathbf{k})$  are related by complex conjugation for a real scalar field. The goal is to find a quantum theory of the free scalar field theory.

For the quantization we proceed in the analogous way to the quantum harmonic oscillator. The Lagrangian density for a free real scalar field, generalized momentum, Hamiltonian density and Poisson brackets<sup>12</sup> are given by

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{m^2}{2} \phi^2 \tag{8.4}$$

$$\pi = \dot{\phi} \tag{8.5}$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 \tag{8.6}$$

$$\{\phi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \delta^{3}(\mathbf{x}-\mathbf{y}) \qquad \{\pi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \{\phi(\mathbf{x},t),\phi(\mathbf{y},t)\} = 0.$$
(8.7)

This is in complete analogy to canonical quantization in quantum mechanics. See App. A for the discussion of the quantum harmonic oscillator. The Hamiltonian is positive definite and thus there is no problem with negative energies. We now quantize the real scalar field and replace the field and its conjugate by hermitian operators. The Poisson brackets are replaced by the canonical (equal-time) commutation relations

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}) \qquad [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = 0.$$
(8.8)

Thus the coefficients  $a(\mathbf{k})$  in Eq. (8.1) become operators

$$\phi(x^{\mu}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}} \left( a(\mathbf{k})e^{-ik_{\mu}x^{\mu}} + a^{\dagger}(\mathbf{k})e^{+ik_{\mu}x^{\mu}} \right)$$
(8.9)

<sup>12</sup>This is a side comment for anyone interested in canonical quantization and the relation between Poisson brackets and commutators. The Poisson brackets for two functionals  $L_i$  are defined as

$$\{L_1, L_2\} = \int d^3x \left[ \frac{\delta L_1}{\delta \phi(x, t)} \frac{\delta L_2}{\delta \pi(x, t)} - \frac{\delta L_1}{\delta \pi(x, t)} \frac{\delta L_2}{\delta \phi(x, t)} \right] , \qquad (8.2)$$

where the functional derivative is defined as

$$\frac{\delta f(t')}{\delta f(t)} = \delta(t - t') . \tag{8.3}$$

with creation operators  $a^{\dagger}(\mathbf{k})$  and annihilation operators  $a(\mathbf{k})$  for every momentum mode. The operators  $a(\mathbf{k})$  and  $a^{\dagger}(\mathbf{k})$  satisfy the commutation relations

$$[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'} \qquad [a(\mathbf{k}), a(\mathbf{k}')] = [a^{\dagger}(\mathbf{k}), a^{\dagger}(\mathbf{k}')] = 0.$$
(8.10)

Exercise: Show this by directly evaluating the commutators. For this you first have to use the field and the conjugate momentum  $\pi = \partial^0 \phi = \dot{\phi}$  to find an expression for  $a(\mathbf{k})$  and  $a^{\dagger}(\mathbf{k})$  in terms of the field and the conjugate momentum  $\pi$ . Similarly we can derive the other commutation relations.

The operators  $a(\mathbf{k})$  and  $a^{\dagger}(\mathbf{k})$  play a similar role to the ladder operators for the quantum harmonic oscillator. There is a ground state which is annihilated by all annihilation operators  $a(\mathbf{k})$ 

$$a(\mathbf{k})\left|0\right\rangle = 0 \quad \forall \mathbf{k} \tag{8.11}$$

and we can define number operators

$$N(\mathbf{k}) = a^{\dagger}(\mathbf{k})a(\mathbf{k}) \tag{8.12}$$

which have as their eigenvalues occupation numbers, i.e. the number of particles with a given 3-momentum  ${f k}$ 

$$n(\mathbf{k}) = 0, 1, 2, \dots \tag{8.13}$$

The set of all quantum states forms a *Fock space*, which consists of a direct sum of Hilbert spaces with  $0, 1, 2, \ldots$  particles.

The Hamiltonian can be rewritten in terms of the number operators as

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left( N(\mathbf{k}) + \frac{1}{2} \right) \tag{8.14}$$

Note that the  $\frac{1}{2}\omega_{\mathbf{k}}$  leads to an infinite vacuum energy in quantum field theory, when summing over all 3-momenta  $\mathbf{k}$ . In the absence of gravity, only energy differences are measured, thus we can define the energy relative to the ground state. The non-vanishing vacuum energy is due to the non-vanishing commutator  $[a(\mathbf{k}), a^{\dagger}(\mathbf{k})] = 1$ . This is formally equivalent to writing all annihilation to the right of the creation operators. It is denoted *normal ordering* and we define the normal ordered product of operators AB by : AB :, e.g.

$$: a(\mathbf{k}_1)a(\mathbf{k}_2)a^{\dagger}(\mathbf{k}_3) := a^{\dagger}(\mathbf{k}_3)a(\mathbf{k}_1)a(\mathbf{k}_2)$$

$$(8.15)$$

Note that the energy is always positive given that the particle number does not become negative. This does not occur given that the norm of the states in the Hilbert space have to be non-negative. Let  $|n(\mathbf{k})\rangle$  be a state with occupation number  $n(\mathbf{k})$  for momentum  $\mathbf{k}$ , then

$$[a(\mathbf{k})|n(\mathbf{k})\rangle]^{\dagger}[a(\mathbf{k})|n(\mathbf{k})\rangle] = \left\langle n(\mathbf{k})|a^{\dagger}(\mathbf{k})a(\mathbf{k})|n(\mathbf{k})\rangle = \langle n(\mathbf{k})|N(\mathbf{k})|n(\mathbf{k})\rangle \simeq n(\mathbf{k})\left\langle n(\mathbf{k})|n(\mathbf{k})\rangle \ge 0.$$

$$(8.16)$$

Let us finally discuss the normalization of states. We normalize the vacuum state to  $1 = \langle 0|0\rangle$ . In order to obtain a Lorentz-invariant normalization of the 1-particle state, we define it as

$$|k\rangle = \sqrt{2k^0 V} a^{\dagger}(\mathbf{k}) \left|0\right\rangle \tag{8.17}$$

and thus find for

$$\langle k|p\rangle = 2V\sqrt{k^0p^0} \left\langle 0|a(\mathbf{k})a^{\dagger}(\mathbf{p})|0\right\rangle = 2V\sqrt{k^0p^0} \left\langle 0|[a(\mathbf{k}),a^{\dagger}(\mathbf{p})]|0\right\rangle = 2k^0 V \delta_{\mathbf{kp}} .$$
(8.18)

This is exactly the normalization which we encountered when deriving the cross section.

## A Quantum harmonic oscillator

Before moving to quantizing a scalar field let us review the quantization of the harmonic oscillator. The quantum harmonic oscillator is defined by the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \tag{A.1}$$

We can directly determine the conjugate momentum, Hamiltonian and Poisson brackets using standard techniques

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{A.2}$$

$$H = p\dot{x} - L = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$$
(A.3)

$$\{x, p\} = 1 \qquad \{p, p\} = \{x, x\} = 0 \tag{A.4}$$

We obtain the quantum harmonic oscillator by replacing x and p by operators  $\hat{x}$  and  $\hat{p} = -i\frac{d}{dx}$  and the Poisson bracket by the commutator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2$$
(A.5)

$$[\hat{x}, \hat{p}] = i$$
  $[\hat{p}, \hat{p}] = [\hat{x}, \hat{x}] = 0$ . (A.6)

In order to solve the quantum harmonic oscillator we want to factorise the Hamiltonian in analogy to the identity

$$u^{2} + v^{2} = (u - iv)(u + iv)$$
 (A.7)

Hence we form two new operators as linear combination of the old ones

$$a = \sqrt{\frac{m\omega}{2}} \left( \hat{x} + i\frac{\hat{p}}{m\omega} \right) \qquad \qquad a^{\dagger} = \sqrt{\frac{m\omega}{2}} \left( \hat{x} - i\frac{\hat{p}}{m\omega} \right) . \tag{A.8}$$

The operators satisfy the following commutation relation

$$[a,a^{\dagger}] = \left[\sqrt{\frac{m\omega}{2}} \left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \sqrt{\frac{m\omega}{2}} \left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\right] = \frac{i}{2}\left([\hat{x}, -\hat{p}] + [\hat{p}, \hat{x}]\right) = 1$$
(A.9)

The Hamiltonian can be rewritten as

$$H = \omega \left( a^{\dagger} a + \frac{1}{2} \right) . \tag{A.10}$$

Exercise: Show by substituting the definitions of a and  $a^{\dagger}$  that it is indeed the Hamiltonian.

Before interpreting the operators  $\hat{a}$  and  $\hat{a}^{\dagger}$  physically, we have to show another commutation relation

$$[H,a] = \left[\omega\left(a^{\dagger}a + \frac{1}{2}\right), a\right] = \omega[a^{\dagger}a, a] = \omega a^{\dagger}[a, a] + \omega[a^{\dagger}, a]a = -\omega a \tag{A.11}$$

or more explicitly

$$[H,a] = Ha - aH \tag{A.12}$$

$$=\omega\left(a^{\dagger}a+\frac{1}{2}\right)a-a\hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right)$$
(A.13)

$$=\omega\left(a^{\dagger}aa - aa^{\dagger}a\right) \tag{A.14}$$

$$= \omega \left( a^{\dagger} a a - \left( a^{\dagger} a + [a, a^{\dagger}] \right) a \right)$$
(A.15)

$$= -\omega a . \tag{A.16}$$

Now given an energy eigenstate  $|E\rangle$  with a given energy E, we can calculate the energy eigenvalue of the states  $a |E\rangle$  as follows

$$H(a|E\rangle) = Ha|E\rangle \tag{A.17}$$

$$= (aH + [H, a]) |E\rangle \tag{A.18}$$

$$= (aE - \omega a) |E\rangle \tag{A.19}$$

$$= (E - \omega)(a | E \rangle) \tag{A.20}$$

Similarly for the operator  $a^{\dagger}$ 

$$[H, a^{\dagger}] = +\omega a^{\dagger} \qquad \qquad H(a^{\dagger} | E \rangle) = (E + \omega)(a^{\dagger} | E \rangle) . \qquad (A.21)$$

Hence the states  $a |E\rangle$ ,  $a^{\dagger} |E\rangle$  are also energy eigenstates with energies  $E \pm \omega$ , respectively. The operators a and  $a^{\dagger}$  transform a state with energy E into a state with energy  $E \pm \omega$ . They are denoted *ladder operators*, more specifically  $a^{\dagger}$  is denoted *raising operator* and *a lowering operator*.

Next we have to find the lowest energy eigenstate or ground state  $|0\rangle$ . Classically we observe that there is a minimum energy of the harmonic oscillator. Hence there has to be a lowest energy eigenstate

$$a\left|0\right\rangle = 0. \tag{A.22}$$

This is called the *ladder termination condition*. The energy of this lowest energy eigenstate is given by

$$H|0\rangle = \omega \left(a^{\dagger}a + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\omega |0\rangle . \qquad (A.23)$$

Note that lowest energy level is not zero as it would be for a classical harmonic oscillator, but  $\frac{1}{2}\omega$ . It is known as *zero point energy* and ultimately due to the non-vanishing commutator of the ladder operators  $[a, a^{\dagger}] = 1$ . The energy of the  $n^{\text{th}}$  state  $|n\rangle$  is given by

$$E_n = \omega \left( n + \frac{1}{2} \right) \,, \tag{A.24}$$

because applying the raising operator  $a^{\dagger} n$  times increases the energy with respect to the lowest energy eigenstate by  $n \times \omega$ . In addition to the ladder operators it is convenient to introduce the number operator,

$$\hat{N} = \hat{a}^{\dagger} \hat{a} , \qquad (A.25)$$



Figure 12: Energy levels of harmonic oscillator. Raising operator  $a^{\dagger}$  increases energy by  $\omega$  and lowering operator a lowers it. Figure taken from Wikipedia.

which counts the energy quanta. It fulfils the following eigenvalue equation

$$N \left| n \right\rangle = n \left| n \right\rangle$$
, (A.26)

where the n in  $|n\rangle$  denotes the number of energy quanta. We can rewrite the Hamiltonian as

$$\hat{H} = \omega \left( \hat{N} + \frac{1}{2} \right) . \tag{A.27}$$

See Fig. 12 for an illustration of the action of the ladder operator on the energy eigenstates. All other energy eigenstates can be constructed from the lowest energy eigenstate using the raising operator. By demanding that all states  $|n\rangle$  are properly normalised,

$$\langle n|n\rangle = 1 , \qquad (A.28)$$

it is possible to show<sup>13</sup> that the raising and lowering operators act on a state  $|n\rangle$ 

$$a|n\rangle = \sqrt{n}|n-1\rangle$$
  $a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$ . (A.29)

Thus we can write the state  $|n\rangle$  as follows

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} \left(a^{\dagger}\right)^{n} |0\rangle \quad , \tag{A.30}$$

where we denoted the lowest energy eigenstate by  $|0\rangle$ . The factor  $1/\sqrt{n!}$  ensures that the states are correctly normalised.

The wave function of the lowest energy eigenstate  $\phi_0(\xi)$  can be determined from the ladder termination condition in Eq. (A.22)

$$0 = a\phi_0(x) = \sqrt{\frac{m\omega}{2}} \left(x + \frac{1}{m\omega}\frac{d}{dx}\right)\phi_0(x) .$$
 (A.31)

<sup>&</sup>lt;sup>13</sup>See the discussion in McIntyre Chap.9.

It is an ODE, which can be solved using standard techniques

$$\phi_0(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega x^2/2}$$
 (A.32)

## **B** Green's function

Consider a differential equation of the form

$$L\psi(x) = f(x) \tag{B.1}$$

with a linear differential operator L = L(x). The Green's function G(x, y) of the differential operator is defined by

$$LG(x,y) = \delta(x-y) . \tag{B.2}$$

It can be used to obtained a general solution for the differential equation Eq. (B.1)

$$\psi(x) = \int dy G(x, y) f(y) . \tag{B.3}$$

This is straightforward to see

$$L\psi(x) = L\left(\int dy G(x,y)f(y)\right) = \int dy LG(x,y)f(y) = \int dy\delta(x-y)f(y) = f(x) .$$
(B.4)

## C Group theory

Symmetries are fundamental ingredients in describing physics. A symmetry transformation is a reversible operation which does not change the physical system. The set of symmetry transformations form a group.

**Definition:** A group G is a set of elements with an operation of multiplication that satisfies the following four properties:

- 1. Closure:  $A, B \in G \Rightarrow AB \in G$
- 2. Associativity:  $A, B, C \in G \Rightarrow (AB)C = A(BC)$
- 3. Identity: there is  $E \in G$  such that AE = EA = A for all  $A \in G$
- 4. Inverse: For all  $A \in G$  there is  $A^{-1} : AA^{-1} = E$

#### **Examples** include

- Trivial group:  $\{E\}$
- Discrete  $Z_2$  group  $\{A, E\} \in Z_2$ ,  $A = A^{-1}$  and thus  $E = AA^{-1} = A^2$
- O(N) the orthogonal group: group of  $n \times n$  orthogonal real matrices

- SO(N) the special orthogonal group: group of  $n \times n$  orthogonal real matrices with unit determinant
- U(1) the unit complex numbers  $\{e^{i\alpha}\}, \alpha \in \mathbb{R}$
- U(N) the unitary group: group of  $n \times n$  complex unitary matrices
- SU(N) the special unitary group: group of  $n \times n$  complex unitary matrices with unit determinant
- Lorentz group SO(3, 1) with its spin group SL(2, C).

The first two examples are so-called discrete finite groups. The remaining examples are Lie groups, which are continuous groups. In fact all of them (apart from the Lorentz group) are compact. U(1), SU(N) and SO(N) are so-called simply Lie groups. The unitary group can be decomposed in terms of the special unitary group and the unit complex numbers:  $U(N) \cong SU(N) \times U(1)$ .

#### C.1 Lie groups

The connected component of Lie group which contains the identity can be described by the underlying Lie algebra. If A is an element in the Lie algebra,  $\exp(\alpha A)$ ,  $\alpha \in \mathbb{R}$  is a one-parameter family of group elements in the Lie group. The vector space of the Lie algebra can be described by the so-called "generators" which form a basis. Thus any element of the Lie algebra (and the connected component of the identity in the Lie group) can be described in terms of those generators. In physics we generally use  $\exp(i\alpha_i T^i)$  with real parameters  $\alpha_i$  and hermitean matrices  $T^i$ . For example the generators of the spin group SU(2) are the Pauli matrices. For SU(3) they are the so-called Gell-Mann matrices. As SU(N) is a unitary group, the generators are hermitean matrices. In physics we typically normalize the generators as  $\operatorname{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ .

#### C.2 Representations

Symmetries leave the system invariant, but transform states  $|\psi\rangle \in \mathcal{H}$  in the Hilbert space. Representations are defined as mappings from the abstract group to linear transformations acting on a vector space which preserve the group structure. For example the group SU(2) can be represented by  $2 \times 2$ special unitary matrices. In particular the generators are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.1}$$

We can however similarly construct a 3-dimensional representation by mapping the three generators of SU(2) to the matrices

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$
(C.2)

### C.3 Special relativity and the Lorentz group

The equivalence of inertial reference frames implies that space is isotropic and homogeneous and time is also homogeneous. It is not possible to distinguish inertial reference frames. In particular, light moves at the same speed in every reference frame F(F'): the distance travelled,  $r^{(\prime)} = ct^{(\prime)}$ . Hence  $0 = (ct)^2 - r^2 = (ct')^2 - r'^2$  and more generally the interval s

$$s^{2} = c^{2}t^{2} - x^{i}x^{i} = c^{2}t'^{2} - x'^{i}x'^{i}$$
(C.3)

with  $r^2 = x^i x^i = \sum_i (x^i)^2$  is the same in all inertial reference frames. Linear transformations which leave the interval s unchanged are known as Lorentz transformations.

We measure time in the same units as distance, i.e. c = 1, and combine both of them in a 4-vector  $x^{\mu}$ ,  $\mu = 0, 1, 2, 3$  with

$$x^0 = t$$
 and  $(x^i) = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$  (C.4)

and thus the interval s is

$$s^{2} = x^{0}x^{0} - x^{i}x^{i} = \eta_{\mu\nu}x^{\mu}x^{\nu}$$
(C.5)

with the metric tensor

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$
(C.6)

From the condition that the interval s is invariant under Lorentz transformations  $x'^{\mu}=\Lambda^{\mu}_{\ \nu}x^{\nu}$ 

$$\eta_{\mu\nu}x^{\prime\mu}x^{\prime\nu} = \eta_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}x^{\rho}x^{\sigma} \equiv \eta_{\rho\sigma}x^{\rho}x^{\sigma} \tag{C.7}$$

it follows that Lorentz transformations satisfy

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} \qquad \Leftrightarrow \eta = \Lambda^T \eta \Lambda \tag{C.8}$$

Taking the determinant we find that

$$\det \Lambda = \begin{cases} 1 & \text{proper Lorentz transformations} \\ -1 & \text{improper Lorentz transformations} \end{cases}$$
(C.9)

and the 00 component yields the condition  $|\Lambda_0^0| \ge 1$ . If  $\Lambda_0^0 \ge 1 (\le -1)$  it is an orthochronous (non-orthochronous) Lorentz transformation. Thus the Lorentz group splits in 4 disconnected components

proper orthochronous  $L^{\uparrow}_{+}$ : det  $\Lambda = 1, \Lambda^{0}_{0} \ge 1$  (C.10)

proper non-orthochronous 
$$L^{\downarrow}_{+}$$
: det  $\Lambda = 1, \Lambda^{0}_{0} \leq -1$  (C.11)

improper orthochronous  $L_{-}^{\uparrow}$ : det  $\Lambda = -1, \Lambda_{0}^{0} \ge 1$  (C.12)

improper non-orthochronous 
$$L_{-}^{\downarrow}$$
: det  $\Lambda = -1, \Lambda_{0}^{0} \leq -1$  (C.13)

The set of events parameterized by  $x^{\mu}$  coordinates together with the metric tensor  $\eta_{\mu\nu}$  defines a space called the Minkowski space-time. Distances are invariant under Lorentz transformations

$$\Delta s^{2} = \eta_{\mu\nu} (x^{\mu} - y^{\mu}) (x^{\nu} - y^{\nu}) \qquad \qquad ds^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \qquad (C.14)$$

Indices are lowered and raised with  $\eta_{\mu\nu}$  and  $\eta^{\mu\nu}$  (the components of the inverse of the metric,  $\eta^{-1}$ . 4-vectors with an upper index  $x^{\mu}$  are called contravariant vectors and with a lower index  $x_{\mu}$  covariant (co-vary with the basis) vectors. Covariant vectors form the dual vector space to the contravariant vectors. A general mixed tensor transforms as

$$T_{\nu_{1}\nu_{2}...\nu_{l}}^{\prime\mu_{1}\mu_{2}...mu_{k}} = \Lambda_{\rho_{1}}^{\mu_{1}}\Lambda_{\rho_{2}}^{\mu_{2}}...\Lambda_{\rho_{k}}^{\mu_{k}}\Lambda_{\nu_{1}}^{\sigma_{1}}\Lambda_{\nu_{2}}^{\sigma_{2}}...\Lambda_{\nu_{l}}^{\sigma_{l}}T_{\sigma_{1}\sigma_{2}...\sigma_{l}}^{\rho_{1}\rho_{2}...\rho_{k}} .$$
(C.15)